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Quasi-particle re-summation and integral gap equation in thermal field theory

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Abstract: A new approach to quantum field theory at finite temperature and density in arbitrary space-time dimension D is developed. We focus mainly on relativistic theories, but the approach applies to non-relativistic ones as well. In this quasi-particle re-summation, the free energy takes the free-field form but with the one-particle energy $\omega(\vec{k})$ replaced by $\varepsilon(\vec{k})$, the latter satisfying a temperature-dependent integral equation with kernel related to a zero temperature form-factor of the trace of stress-energy tensor. For 2D integrable theories the approach reduces to the thermodynamic Bethe ansatz. For relativistic theories, a thermal c-function $C_{\rm qs}(T)$ is defined for any D based on the coefficient of the black body radiation formula. Thermodynamical constraints on it's flow are presented, showing that it can violate a "c-theorem" even in 2D. At a fixed point $C_{\rm qs}$ is a function of thermal gap parameters which generalizes Roger's dilogarithm to higher dimensions. This points to a strategy for classifying rational theories based on "polylogarithmic ladders" in mathematics, and many examples are worked out. Other applications are discussed, including the free energy of anyons in 2D and 3D, phase transitions with a chemical potential, and the equation of state for cosmological dark energy. For the latter we show that $-1 < w < -1/3$.

Keywords: Anyons, Conformal and W Symmetry, Thermal Field Theory, Bethe Ansatz.

Contents

1. Introduction

Quantum field theory at finite temperature and density was developed long ago mainly to deal with condensed matter systems at low temperatures. The subject was later taken up in the relativistic context where the interest was mainly in high temperatures [1, 2], with the aim for example to study hot QCD [4]. In both of these contexts the main computational tool is the beautiful but sometimes awkward sums over Matsubara frequencies in perturbation theory. It was later realized that to capture some important physics, such as the gluon relaxation rate, a re-summation scheme was necessary, beyond the re-summation of "ring diagrams", at least to re-sum the hard thermal loops [3]. In spite of this important progress, finite temperature field theory remains difficult, especially when dealing with phase transitions, and it is worthwhile to explore alternative methods. In the condensed matter context other kinds of self-consistent re-summations are often used, for example in the finite temperature gap equation of the BCS theory of superconductivity. The main purpose of this paper is to develop a new approach to finite temperature field theory in any dimension D in a similar but essentially different spirit. Because the object we are dealing with, i.e. a quantum field functional integral in finite size geometry, is so general, there are many possible applications; we will touch on a number of them, such as central charges at fixed points, classical statistical mechanics of Ising-like models in any dimension, anyons and cosmology.

In sections 2 and 3 we introduce the usual quantities of quantum statistical mechanics, the free energy, pressure, entropy, etc. and express them as one-point functions of the stress-energy tensor. It is meaningful to express everything in terms of a temperature dependent c-function $C_{\rm qs}(T)$ which at a renormalization group (RG) fixed point becomes the coefficient in the black body formula. (The subscript $_{\rm qs}$ stands for *quantum statistical* and is meant emphasize the distinction with other c-functions.) We show that in 4D, this thermal c-function is different from other c-functions based on conformal anomalies [18]. We also argue that thermodynamic principles, such as positivity of the entropy, do not lead to any "c-theorem's" for $C_{\rm qs}$.

$C_{\rm qs}(T)$ is expected to be important for equations of state in the thermal universe. In section 4 we relate the dark energy parameter w to a temperature derivative of $\log C_{\rm qs}$. Using the entropy bounds on $C_{\rm qs}$ we argue that for negative pressure $-1 < w < -1/3$.

The main workings of this paper begin in section 5. The idea is to assume that the partition function can be approximated by the standard free field form, but with the one particle energies $\omega_{\mathbf{k}}$ replaced by a quasi-particle energy $\varepsilon(\mathbf{k})$. A saddle point approximation to the partition function then leads to an integral gap equation satisfied by ε. The kernel in the integral equation is then related to zero temperature two-particle form factor of the trace of the stress-energy tensor T^μ_μ. This approach is modeled after the thermodynamic Bethe ansatz (TBA) [34, 35, 32], which is exact for the 2D integrable quantum field theories. For the non-integrable theories in D dimensions we don't expect the results to be exact, but rather to be a potentially useful approximation. In section 6 we check that to lowest order our formalism gives the same result as the two-loop Matsubara formalism for the scalar ϕ^4 theory.

At an RG fixed point, $C_{\rm qs}$ becomes a constant $c_{\rm qs}$ which is an important characteristic of the conformal field theory. There are no known methods for computing $c_{\rm qs}$ for interacting fixed points in dimension $D > 2$. As a step in this direction, in section 7 we describe how to compute $c_{\rm qs}$ in our formalism. The basic mechanism, which we believe to be general, is that at a fixed point the quasi-particle energy ε develops a gap Δ that depends only on the temperature $\Delta = gT$. The gap parameter g can be computed from an algebraic limit of the gap equation, and $c_{\rm qs}$ is a function only of g which is a higher dimensional generalization of Roger's dilogarithm. In this way a fixed point theory is deconstructed into the thermal gap data, which depends only on *interaction parameters* that are an integral of the form factor of T^{μ}_{μ}.

The classification of theories in any dimension with rational $c_{\rm qs}$ is shown to be very closely related to the construction of polylogarithmic ladders in mathematics [43], in particular classical analysis and number theory. Various aspects of this are explored in sections 9–11. Given a rational $c_{\rm qs}$ that can be obtained from the higher dimensional polylogarithmic expressions of section 7, one can work back-wards by reconstructing the algebraic gap equation. The latter gives some information on the interaction parameters. For integrable theories in 2D, due to the tight constraints of integrability, one can in many cases entirely reconstruct the S-matrix of the theory. In higher dimensions we do not as yet have the tools necessary to complete the reconstruction of the fixed point theory. However we believe our analysis points to the existence of interesting conformal theories in higher dimensions and we can at least reconstruct some of their properties, such as particle spectrum and interaction parameters.

An interesting example worked out in detail is that of a single bosonic particle with gap parameter $g = \log 2$. The polylogarithmic identities leading to $c_{\rm qs} = 1/2$ in 2D and $c_{\rm qs} = 7/8$ in 3D where known to Euler and Landen. Though $c_{\rm qs} = 1/2$ is the central charge of a free fermion in 2D, the physical meaning of our bosonic theory and its possible connection to the Ising model will remain unanswered. The reason it has not been studied before is that previously nobody has been able to make sense of a *bosonic* quantum statistical mechanics in the TBA framework [47].

In section 12 it is shown how a purely imaginary chemical potential can describe fractional statistics particles. In this construction it is necessary to work with gap parameters in the complex plane $z = e^{-g}$. For free "anyons" the gap parameters z are on the unit circle. In 2D the free anyons correspond to $\beta - \gamma$ systems in a continuous sub-regime of the spin $\sigma = \vartheta/2\pi$ of the particles where $0 < \vartheta < 2\pi$. When the spin is rational, so is $c_{\rm qs}$. Extending this construction to 3D we propose a formula for the free energy of free anyons. Here $c_{\rm qs}$ is only rational for special values of the statistics parameter such as $\vartheta/2\pi = 1/3$.

In section 13 we reintroduce the chemical potential and analyze the phase structure. This leads to general statements concerning the existence of fixed points depending on the interaction strengths and chemical potential.

2. Free energy and the stress-energy tensor

In this section we introduce the main physical quantities that are the subject of this paper; we mainly collect some well-known formulas. From the most general point of view, we will be dealing with a euclidean functional integral of fields in $D = d+1$ dimensions where one of the dimensions is compactified into a circle of circumference β. This object arises in a variety of physical contexts. It is the finite temperature T partition function of a quantum many-body system or quantum field theory in d spacial dimensions where the extra dimension is the euclidean time and $\beta = 1/T$. It can also represent a partition function of classical statistical mechanics in D spacial dimensions with one finite size direction of length β. We will mainly be using the language of quantum statistical mechanics. Most formulas will be expressed in terms of β, but some formulas will be expressed in terms of the temperature T when this makes the physical context more transparent. Some formulas will be expressed in terms of D, others in terms of d, depending on what is more convenient.

The partition function is

$$Z(\beta,\mu) = \mathrm{Tr}\ e^{-\beta(H-\mu\widehat{N})}\,, \tag{2.1}$$

where H is the many-body hamiltonian, $\widehat{N}$ the particle number operator, and μ the chemical potential. The free energy density, $\mathcal{F} = F/V$, is defined as:

$$\mathcal{F} = -\frac{1}{\beta V}\log Z\,, \tag{2.2}$$

where V is the d-dimensional spacial volume. The energy density $\mathcal{E}$ per volume and pressure p are defined as

$$\mathcal{E} = -\frac{1}{V}\frac{\partial \log Z}{\partial\beta} = \frac{\partial(\beta\mathcal{F})}{\partial\beta}\,, \qquad p = \frac{1}{\beta}\frac{\partial \log Z}{\partial V} = -\mathcal{F}\,. \tag{2.3}$$

The above quantities are related to the stress-energy tensor of the theory $T_{\mu\nu}(\mathbf{x})$, $\mu,\nu = 0,1,\ldots,d$. The hamiltonian H and momentum operators $\vec{P}$ are given by

$$H = \int d\mathbf{x}\ T_{00}(\mathbf{x})\,, \qquad P_i = \int d\mathbf{x}\ T_{0i}(\mathbf{x})\,, \tag{2.4}$$

where $d\mathbf{x} \equiv d^d\vec{x}$. The energy density and pressure are related to finite temperature one-point functions of $T_{\mu\nu}$:

$$\langle T_{00}\rangle_\beta = \mathcal{E}\,, \qquad \langle T_{ii}\rangle_\beta = p \tag{2.5}$$

(no sum on i), where $\langle * \rangle_\beta$ denotes the finite temperature correlation function. (The one-point functions are independent of $\mathbf{x}$ by translation invariance.) The trace of the stress-energy tensor, $T^\mu_\mu \equiv \sum_{\mu=0}^d T^\mu_\mu$, can thus be related to the free energy:

$$\langle T^\mu_\mu\rangle_\beta = (\beta\partial_\beta + d + 1)\ \mathcal{F}\,. \tag{2.6}$$

In the sequel we will also deal with the entropy density $\mathcal{S}$ and number density n:

$$\mathcal{S} = \beta^2\frac{\partial\mathcal{F}}{\partial\beta} = \beta(\mathcal{E}-\mathcal{F})\,, \qquad n = -\frac{\partial\mathcal{F}}{\partial\mu}\,. \tag{2.7}$$

The first equation is just the first law of the thermodynamics.

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3. Quantum statistical c-functions in D dimensions

For a free gas of bosons or fermions, the partition function is well-known in the limit of infinite volume V:

$$\log Z = \mp V \int \frac{d\mathbf{k}}{(2\pi)^d} \, \log \left(1 \mp e^{-\beta\omega_{\mathbf{k}}}\right) , \tag{3.1}$$

where the upper (lower) sign corresponds to bosons (fermions), $d\mathbf{k} \equiv d^d\vec{k}$, and $\omega_{\mathbf{k}}$ is the single particle energy as a function of the momentum $\mathbf{k} \equiv \vec{k}$, which need not be relativistic. For a massless relativistic theory with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$, the integrals for the basic thermodynamic quantities can be done analytically. For example in 4D, the energy density is the well-known black body formula:

$$\mathcal{E} = c_{\mathrm{qs}} \frac{\pi^2}{30} T^4 , \tag{3.2}$$

where $c_{\mathrm{qs}} = 1$ for a free boson and $7/8$ for a free fermion. One can also relate the entropy S and the number of particles $N = nV$:

$$S = \mathrm{e}_{\mp} \, N , \tag{3.3}$$

where e_- (e_+) is the entropy per particle for bosons (fermions):

$$\mathrm{e}_- = \frac{(d+1)\zeta(d+1)}{\zeta(d)} , \qquad \mathrm{e}_+ = \mathrm{e}_- \left(\frac{2^d - 1}{2^d - 2}\right) \tag{3.4}$$

and $\zeta(z)$ is the Riemann zeta function. Finally the ideal gas law reads:

$$pV = \frac{\mathrm{e}_{\mp}}{d+1} NT . \tag{3.5}$$

One question that will be addressed in this paper is how interactions can change the coefficient c_{qs} in eq. (3.2) even at a fixed point. This leads us to introduce a quantum statistical c-function, $C_{\mathrm{qs}}(T)$, which depends on temperature, and at a fixed point becomes in 4D for example c_{qs} in eq. (3.2). This quantity can be defined in any dimension. If the theory is relativistic and conformally invariant, the trace of the stress tensor vanishes $\langle T^\mu_\mu \rangle_\beta = 0$. Equation (2.6) then implies that $\mathcal{F} = \mathrm{const.}/\beta^{d+1}$. This leads to a natural definition of a thermal c-function based on quantum statistical mechanics:

$$\mathcal{F} = -\frac{a_d}{\beta^{d+1}} \, C_{\mathrm{qs}}(\beta, \lambda) , \tag{3.6}$$

where a_d is a normalization constant. The function C_{qs} depends on β and the couplings and masses of the theory, here denoted generically as λ. Because the definition eq. (3.6), which is essentially motivated simply by dimensional analysis, is so natural, it has previously appeared in a number of works [5–7, 9–12]. (In [5] the finite size, rather than finite temperature, version was considered.) This paper is not so much concerned with proving or disproving c-theorems, but rather with developing a new framework for computing C_{qs}. Nevertheless, some remarks on c-theorems can be found later in this section.

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The trace of the stress-energy tensor obeys renormalization group equations (RG) since it is a one-point function of a quantum field theory on a finite size geometry, the length scale being β. Thus:

$$\Theta(e^s\beta, \lambda(l)) = e^{-Ds}\Theta(\beta, \lambda(l+s)) , \qquad \Theta(\beta, \lambda) \equiv \langle T^\mu_\mu \rangle_\beta . \tag{3.7}$$

Above, l is the log of the length scale, and the l dependence of λ is determined by the RG beta-functions: $d\lambda/dl = \dot{\lambda}(\lambda)$. The above equation shows that $C_{\rm qs}(\beta)$ as a function of β tracks the RG flow. Using eq. (2.6) one has:

$$\Theta(\beta) = \mathcal{E} - dp = a_d T^{d+2} \, \frac{\partial C_{\rm qs}}{\partial T} . \tag{3.8}$$

Thus at an RG fixed point $\Theta = 0$, $C_{\rm qs}$ is a constant independent of β, and is an important characteristic of the fixed point theory. We will use the lower case $c_{\rm qs}$ to denote the β independent fixed point value of $C_{\rm qs}$.

The entropy and energy densities in terms of $C_{\rm qs}$ are:

$$\mathcal{S} = \frac{a_d}{\beta^d} \left((d+1) - \beta\partial_\beta\right) C_{\rm qs} \tag{3.9}$$

$$\mathcal{E} = \frac{a_d}{\beta^d} \left(d - \beta\partial_\beta\right) C_{\rm qs} . \tag{3.10}$$

At a fixed point one has:

$$\mathcal{E} = d \frac{a_d}{\beta^{d+1}} \, c_{\rm qs} , \qquad \mathcal{S} = (d+1) \frac{a_d}{\beta^d} \, c_{\rm qs} \tag{3.11}$$

so that $\mathcal{E}/\mathcal{S} = \frac{d}{d+1} T$.

We chose a_d so that $c_{\rm qs} = 1$ for a free, massless relativistic boson. As we will show below, this leads to:

$$a_{D-1} \equiv \pi^{-D/2} \Gamma(D/2) \zeta(D) , \tag{3.12}$$

where Γ is the gamma-function and as before ζ the Riemann zeta-function. When D is even, $\zeta(D)$ is rational up to powers of π: $\zeta(D) = 2^{D-1}\pi^D |B_D|/D!$, where B_D are rational Bernoulli numbers. This is not the case for D odd; in fact it has only been proven relatively recently that $\zeta(3)$ is irrational. We checked numerically that $\zeta(D)/\pi^D$ for D odd doesn't appear to be rational. As we explain below, this appears to be related to the fact that there is no conformal anomaly for D odd. For $D = 2, 3, 4$ the expression for the free energy then becomes:

$$\begin{aligned} \mathcal{F} &= -C_{\rm qs}(T) \, \frac{\pi T^2}{6} \qquad (D=2) \\ &= -C_{\rm qs}(T) \, \frac{\zeta(3) T^3}{2\pi} \qquad (D=3) \\ &= -C_{\rm qs}(T) \frac{\pi^2 T^4}{90} \qquad (D=4) . \end{aligned} \tag{3.13}$$

Since in recent years there has been renewed interest in possible generalizations of Zamolodchikov's c-theorem [13] to higher dimensions [18 – 21], we now discuss the relation of $c_{\rm qs}$ to other proposals in higher dimensions. Let us first review the success story in

$D = 2$. In a conformally invariant theory, c can be defined by the leading singularity of the operator product expansion of two stress-energy tensors, or equivalently the central extension in the Virasoro algebra [14]. The same central charge c appears as a conformal anomaly in the conformal field theory:

$$\langle T^{\mu}_{\mu} \rangle = -\frac{c}{48\pi} R \,, \tag{3.14}$$

where R is the scalar gravitational curvature. Using the above relation one can show that c governs the finite size effects [16, 17] of the partition function and thus at a fixed point c is the same as $c_{\rm qs}$.

For the purpose of studying the RG flow between fixed points there are two known alternative approaches to defining a scale dependent c that tracks the flow and equals the Virasoro central charge at the fixed point. One is to define $C_{\rm qs}(\beta)$ as above [35]. The other is the content of the c-theorem where one defines a function $c_z(L)$ from the two-point functions of $T_{\mu\nu}$ at zero temperature but at finite separation L. The c-theorem states that $c_z(L)$ monotonically decreases as a function of the increasing length scale L. There is sometimes the misconception that $C_{\rm qs}$ and c_z are essentially the same. This misconception has led to the expectation that $C_{\rm qs}$ also always decreases, even in 2D. However these two c's are potentially very different, as one is related to a one-point function at finite temperature, the other a two-point function at zero temperature. The thermodynamic foundation for $C_{\rm qs}$ actually makes it a much richer physical quantity, and obviously one that can be defined in any dimension.

Though some discussions of the c-theorem vaguely invoke notions in statistical physics like irreversibility and loss of information, in reality c_z has no statistical mechanical meaning. The intuitive understanding is more accurately that c_z is a measure of the massless states of the theory, and since massive states decouple at low energies, c_z decreases [15]. Ironically, whereas $C_{\rm qs}(\beta)$ does have a thermodynamic meaning, no theorem has ever been proven for it. Indeed, though $C_{\rm qs}$ is known to decrease for theories with UV and IR fixed points, it can actually oscillate in physically sound theories with RG limit cycles [23, 24].

As eq. (3.8) obviously shows, a c-theorem for $C_{\rm qs}$ follows from positivity of Θ: if $\Theta > 0$, then $\partial C_{\rm qs}/\partial T > 0$, i.e. $C_{\rm qs}$ increases with temperature. Let us refer to the above condition as Θ-positivity. In [11] many interesting examples were presented that exhibit the expected property ${c_{\rm qs}}^{IR} < {c_{\rm qs}}^{UV}$; however examples where $C_{\rm qs}$ does not decrease monotonically with decreasing temperature were also discussed.

A general manner in which Θ-positivity can be violated is as follows [11, 24]. Suppose a quantum field theory is described by a conformally invariant action $S_{\rm cft}$ perturbed by operators $\mathcal{O}^A$:

$$S = S_{\rm cft} + \sum_A \int \frac{d^2x}{2\pi} \, \lambda_A \mathcal{O}^A \,, \tag{3.15}$$

where λ_A are positive couplings. Let $\dot{\lambda}_A$ be the beta function for λ_A and Γ_A the scaling dimension (including anomalous) of $\mathcal{O}^A$. Then the beta functions $\dot{\lambda}_A = d\lambda_A/d\log L$, where L is a length scale, to lowest order are

$$\dot{\lambda}_A = (D - \Gamma_A)\lambda_A + O(\lambda^2) \,. \tag{3.16}$$

The main point is that the trace of the stress-energy tensor receives quantum corrections, and since it must be zero at a fixed point where the beta functions are zero, one must have:

$$T^{\mu}_{\mu} = \sum_{A} \dot{\lambda}_A(\lambda)\, \mathcal{O}^A \,. \tag{3.17}$$

The above formula is well known and is easily verified in 2D to lowest order in conformal perturbation theory [25]. Consider first a theory that can be formulated as a perturbation of an ultra-violet fixed point by relevant operators, which implies $\Gamma_A < D$. Then the beta-functions are positive to lowest order. Furthermore, for relevant perturbations, because of the anomalous dimensions of the couplings, there is often no higher order corrections to the beta functions since higher powers of λ_A do not have the right dimension. So in this situation, T^{μ}_{μ} is generally positive and the c-theorem holds.

The above arguments clearly point to the way in which the c-theorem can be violated. If the $\mathcal{O}^A$ are marginal, $\Gamma_A = D$, and the beta functions start at $O(\lambda^2)$, and there are no general constraints on the sign of the beta function. This was suggested as the origin of the violation of the c-theorem for the 2D models with limit-cycles in the RG [23, 24], though the issue is not entirely resolved.

What constraints on the flow of $C_{\rm qs}$ follow from the most basic thermodynamic principles? Note that the entropy itself, and not its change, already depends on the flow of $C_{\rm qs}$, eq. (3.9). One constraint is that the entropy density should be positive. Since the pressure is proportional to $C_{\rm qs}$, there are two cases depending on the sign of $C_{\rm qs}$. Using eq. (3.9) and (3.10) (in terms of temperature T):

$$\begin{aligned} \textbf{positive pressure}: \qquad & \mathcal{S} > 0 \qquad \Longrightarrow \qquad \frac{\partial \log C_{\rm qs}}{\partial \log T} > -d-1 \\ & \mathcal{E} > 0 \qquad \Longrightarrow \qquad \frac{\partial \log C_{\rm qs}}{\partial \log T} > -d \end{aligned} \tag{3.18}$$

$$\begin{aligned} \textbf{negative pressure}: \qquad & \mathcal{S} > 0 \qquad \Longrightarrow \qquad \frac{\partial \log C_{\rm qs}}{\partial \log T} < -d-1 \\ & \mathcal{E} > 0 \qquad \Longrightarrow \qquad \frac{\partial \log C_{\rm qs}}{\partial \log T} < -d\,. \end{aligned} \tag{3.19}$$

Note that in the positive pressure case, a positive entropy could have a positive or negative energy density. On the other hand in the negative pressure case, a positive entropy necessarily implies a positive energy density.

Let us now turn to the conformal anomalies in higher dimensions. In D dimensions T^{μ}_{μ} has scaling dimension D and the curvature R has dimension 2. This already indicates that there is no conformal anomaly for D odd since no power of the curvature has dimension D. As mentioned above, this appears to be reflected in the formulas eq. (3.13) by the irrationality of $\zeta(D)$ up to geometrical powers of π for D odd. Again the quantity $C_{\rm qs}$ presents itself as the most natural analog to c for all even and odd dimensions.

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For $D = 4$ the conformal anomaly reads [22, 21]

$$\langle T^\mu_\mu \rangle = \frac{1}{1920\pi^2} \left(\widehat{c} W^2 - a \widetilde{R}^2 \right), \tag{3.20}$$

where W is the Weyl tensor and $\widetilde{R}$ the dual curvature. (We have rescaled $\widehat{c}, a$ so that $\widehat{c} = 1$ for a free boson.) Cardy conjectured an a-theorem for the anomaly a, and recently there have been many interesting examples of supersymmetric gauge theory supporting the conjecture [21, 19, 20]. Many of these examples rely on Seiberg duality. It should be emphasized that the above equation is valid only at a fixed point, and thus does not provide a definition of scale-dependent $\widehat{c}, a$ that interpolate between fixed points. Since $\widehat{c}$ and a are related to one-point functions of T^μ_μ, they are analogous to $c_{\rm qs}$ rather than c_z. However one can easily show that no linear combination of $\widehat{c}, a$ equals $c_{\rm qs}$. Consider a free theory consisting of N_0 scalars, $N_{1/2}$ spin 1/2 Majorana fermions, and N_1 spin 1 gauge bosons. The anomalies are known [22, 21]:

$$\begin{aligned} \widehat{c} &= N_0 + 3N_{1/2} + 12N_1 \\ a &= \frac{(2N_0 + 11N_{1/2} + 124N_1)}{6} \,. \end{aligned} \tag{3.21}$$

On the other hand $c_{\rm qs}$ only depends on the statistics of the particles and their number of degrees of freedom, which are 2 helicities for the fermions and 2 polarizations for the gauge bosons. Using $c_{\rm qs} = 1$ for a boson and $c_{\rm qs} = 7/8$ for a fermion:

$$c_{\rm qs} = N_0 + \frac{7}{4} N_{1/2} + 2N_1 \,. \tag{3.22}$$

Noting that $3(\widehat{c} - a)/2 = N_0 + 7N_{1/2}/4 - 13N_1$, one concludes that in a free theory of only scalars and spin 1/2 particles, $c_{\rm qs} = 3(\widehat{c} - a)/2$, however for higher spin particles there is no linear combination of $\widehat{c}$ and a that equals $c_{\rm qs}$.

Since the equation (3.20) involves the curvature, the anomalies $\widehat{c}, a$ can determine some finite size effects. However unlike the $D = 2$ situation, $\widehat{c}, a$ apparently do not govern the finite size effects having to do with finite temperature. Indeed, lattice simulations of finite temperature pure QCD observe the behavior eq. (3.2) at high temperature with $c_{\rm qs} = 16$ which is consistent with an SU(3) gauge theory with 8 gauge bosons [26]. The fact that $c_{\rm qs}$ is here given by the free field values is an indication of asymptotic freedom of QCD [4].

4. Quantum statistical c-function in cosmology

The free field fixed point values of $C_{\rm qs}$ are used extensively in studies of the thermodynamics of the hot early universe. For instance the energy density is taken to be given by the black body formula (3.2) for a radiation dominated universe. This is clearly an approximation in a number of respects. First of all, even at a fixed point, the interactions can change the value of $c_{\rm qs}$, unless the theory is asymptotically free. This will be the subject of section 7 of this paper. Secondly, the universe is probably not at a fixed point. Our comments on cosmology will be confined to this section, where we primarily phrase some

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cosmological properties in terms of $C_{\rm qs}$ and use this to place bounds dark energy that is not a cosmological constant. In other words, we place bounds on dark energy that is hypothetically made up of particles.

There is growing observational evidence for an acceleration of the expansion of the universe. The cause of this phenomenon is commonly referred to as dark energy. (See for instance, [27–29].) Dark energy is usually characterized by the equation of state parameter w:

$$w \equiv \frac{p}{\mathcal{E}} . \tag{4.1}$$

One of the Friedmann equations expresses the acceleration in terms of $\mathcal{E}, p$:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\mathcal{E} + dp) = -\frac{4\pi G}{3}\left(\frac{1}{w} + d\right) p , \tag{4.2}$$

where throughout this section $d = 3$ and R is the cosmic scale factor. Accelerated expansion, i.e. dark energy, corresponds to $\ddot{R}/R > 0$. If the pressure is positive, then dark energy requires $-1/d < w < 0$ which corresponds to a negative energy density. If on the other hand the pressure is negative, dark energy requires $w < -1/d$ with positive energy density, or $w > 0$ but with negative energy density.

The parameter w can be expressed in terms of $C_{\rm qs}$:

$$\frac{1}{w} = d + \frac{\partial \log C_{\rm qs}}{\partial \log T} . \tag{4.3}$$

At a fixed point, $w = 1/d$ regardless of the value of $c_{\rm qs}$. However away from a fixed point, it can be different. Already one sees that if dark energy is matter, it is currently not at an RG fixed point. The thermodynamic bounds coming from the requirement of positive entropy, eqs. (3.18), (3.19), place bounds on w:

$$\begin{array}{lll} \text{positive pressure :} & \mathcal{S} > 0 \Longrightarrow & w < -1 \quad \text{or} \quad w > 0 \\ \text{negative pressure :} & \mathcal{S} > 0 \Longrightarrow & -1 < w < 0 . \end{array} \tag{4.4}$$

Thus one sees that positive pressure dark energy, which required $-1/d < w < 0$, is inconsistent with positive entropy. More importantly, taking into account the entropy bounds, negative pressure dark energy necessarily corresponds to $-1 < w < -1/d$.

The standard candidate for dark energy is a cosmological constant Λ, where the stress tensor $T^{\mu\nu} \propto \Lambda g^{\mu\nu}$. In Minkowski space, this can correspond to the stress-energy tensor of vacuum energy. From the Lorentz structure of the flat space metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, this necessarily implies $p = -\mathcal{E}$ and $w = -1$. Interestingly, the lower limit $w = -1$, which we obtained from thermodynamic principles, precisely mimics a cosmological constant.

If dark energy is not a cosmological constant, then in terms of $C_{\rm qs}$ the most likely scenario for dark energy is that the universe is currently at a temperature where $C_{\rm qs}$ is negative with $\partial C_{\rm qs}/dT > 0$, i.e. $C_{\rm qs}$ is decreasing as the temperature continues to decrease.

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The fate of the universe then depends on whether there is an RG fixed point at zero temperature. If there is a fixed point, then as explained above, w will eventually become positive and equal to $1/3$.

As we will see, obtaining a negative $C_{\rm qs}$ is not so straightforward in our formalism. One can study this by examining fixed point values $c_{\rm qs}$ analytically and see what is required for $c_{\rm qs}$ to be negative, since $C_{\rm qs}$ is bound to be negative in the vicinity of the fixed point. As we will show in section 13, this seems to require an imaginary part to the chemical potential. One interpretation of an imaginary chemical potential in 2D and 3D will be given in section 12, i.e. as corresponding to fractional statistics particles. Analogous Hopf terms in 4D would require $Im(\mu) = i\pi T$ since there are only bosons and fermions in 4D; one example is skyrmions [30]. But there may be other interpretations to an imaginary part, for example a relaxation rate.

5. Quasi-particle re-summation

This section is rather technical, so let us begin by drawing attention to the main results: eq. (5.9) is the desired form of the partition function, where ε satisfies the integral gap equation (5.7). The kernel appearing in this integral equation is given by the form factor eq. (5.32).

Interacting field theories, even at finite temperature, normally have a particle description. In many complex condensed matter systems one is accustomed to dealing with quasi-particle excitations of definite statistics, for example in Landau's theory of a Fermi liquid for metals. The free particle form of $\log Z$, eq. (3.1) is appealing since it incorporates the bosonic or fermionic statistics of the particles. The eq. (3.1) can of course be obtained from a functional integral in finite size which involves the Matsubara frequencies, but only with some effort; the simplicity of the final answer suggests there is a simpler, particle description. Based on these observations, we formulate a quasi-particle description of Z with the following properties. We suppose that the free particle form eq. (3.1) continues to hold but with $\omega_{\mathbf{k}}$ replaced by a quasi-particle energy $\varepsilon(\mathbf{k})$. All the effects of the interactions, and additional finite temperature effects are all contained in ε.

There are no general arguments showing that finite temperature field theory can be organized in this fashion. The formulation should be viewed as a potentially useful approximation that can capture some physical effects in a transparent way in comparison with other approaches. However for 2D integrable theories this formulation is exact and known as the thermodynamic Bethe ansatz, and this certainly provided some motivation and insights.

We first describe a particle description of the partition function. To proceed, introduce a density of particles $\rho(\mathbf{k})$ so that $\rho(\mathbf{k})d\mathbf{k}$ represents the number of particles with momentum between $\mathbf{k}$ and $\mathbf{k}+d\mathbf{k}$. Similarly introduce the level density $\rho_l(\mathbf{k})$ which counts the available quantum states. In a free theory ρ_l just counts the states for particles in a box of volume V and $\rho_l = V/(2\pi)^d$. In an interacting theory the densities ρ and ρ_l are clearly not independent: when a new particle is added it affects the available levels because of the interactions. For integrable theories in 2D the quantization condition on allowed momenta leads to a relation between ρ and ρ_l involving the 2-particle factorized S-matrix [34, 35]. (See section 10.2.) It is not clear that this kind of quantization condition can actually be formulated in higher dimensions. So let us for now simply assume that there is a linear relation between the densities that depends on the momenta. This can generally be written in terms of a temperature independent Fredholm integral operator (kernel) $\mathbf{G}$:

$$\rho_l(\mathbf{k}) = \frac{V}{(2\pi)^d} + \int \frac{d\mathbf{k}'}{(2\pi)^d} \, \rho(\mathbf{k}') G(\mathbf{k}', \mathbf{k}) \frac{1}{\omega_{\mathbf{k}}} \,. \tag{5.1}$$

The kernel $\mathbf{G}$ will be determined below. However as we now show, *any* choice of $\mathbf{G}$ leads to the desired form of the partition function. Retaining the chemical potential, one has

$$\log Z = -\beta \int d\mathbf{k} \, (\omega_{\mathbf{k}} - \mu) \rho(\mathbf{k}) + \$ \,, \tag{5.2}$$

where $\$$ is the entropy. The entropy depends on the statistics of the particles:

$$\$ = \pm \int d\mathbf{k} \left((\rho_l \pm \rho) \log(\rho_l \pm \rho) - \rho_l \log \rho_l \mp \rho \log \rho \right), \tag{5.3}$$

where the upper/lower sign is for boson/fermion statistics. (In the subsequent formulas the boson case will always be the upper sign.) We now look for a saddle point approximation to Z. Varying with respect to both densities:

$$\delta \log Z = \left(-\beta(\omega - \mu) + \log\left(\frac{\rho_l}{\rho} \pm 1 \right) \right) \delta\rho \pm \log\left(1 \pm \frac{\rho}{\rho_l} \right) \delta\rho_l \,. \tag{5.4}$$

The saddle point is subject to the constraint eq. (5.1) relating ρ, ρ_l. The saddle point equation $\delta \log Z = 0$ then has a simple form. Define the filling fractions

$$f \equiv \frac{\rho}{\rho_l} \tag{5.5}$$

and parameterize them as follows:

$$f_{\mp} = \frac{1}{e^{\beta\varepsilon} \mp 1} \,. \tag{5.6}$$

Then the saddle point equation becomes a non-linear integral equation involving $\mathbf{G}$:

$$\varepsilon(\mathbf{k}) = (\omega_{\mathbf{k}} - \mu) \pm \frac{1}{\beta} \int \underline{\underline{d\mathbf{k}'}} \; G(\mathbf{k}, \mathbf{k}') \; \log\left(1 \mp e^{-\beta\varepsilon(\mathbf{k}')} \right), \tag{5.7}$$

where we have defined:

$$\underline{\underline{d\mathbf{k}}} \equiv \frac{d\mathbf{k}}{(2\pi)^d \omega_{\mathbf{k}}} \tag{5.8}$$

which is Lorentz invariant when the theory is relativistic. Plugging this back into $\log Z$ one finds that it has the desired form:

$$\log Z = \mp V \int \frac{d\mathbf{k}}{(2\pi)^d} \; \log\left(1 \mp e^{-\beta\varepsilon(\mathbf{k})} \right). \tag{5.9}$$

We will refer to the equation (5.7) as the *integral gap equation*.

The fundamental thermodynamic quantities, obtained by differentiation, have simple expressions:

$$\mathcal{E} = \int \frac{d\mathbf{k}}{(2\pi)^d} \; f_{\mp}(\mathbf{k}) \; (1+\beta\partial_\beta)\varepsilon \tag{5.10}$$

$$n = -\int \frac{d\mathbf{k}}{(2\pi)^d} \; f_{\mp}(\mathbf{k}) \; \frac{\partial\varepsilon}{\partial\mu} \,. \tag{5.11}$$

Also, using $\nabla_{\mathbf{k}} \cdot \mathbf{k} = d$, one has for the pressure

$$p = \frac{1}{d}\int \frac{d\mathbf{k}}{(2\pi)^d} \; f_{\mp}(\mathbf{k}) \; \mathbf{k}\cdot \nabla_{\mathbf{k}}\, \varepsilon(\mathbf{k}) \,. \tag{5.12}$$

Using the integral equation satisfied by ε, one can obtain an integral series representation for the above quantities. Introduce the shorthand notation

$$(\mathbf{G} * \phi)(\mathbf{k}) \equiv \int \underline{\underline{d\mathbf{k}'}} \; G(\mathbf{k},\mathbf{k}')\phi(\mathbf{k}') \tag{5.13}$$

for an arbitrary function $\phi(\mathbf{k})$. (Where there is no chance for confusion we will sometimes drop the $*$.) Then using the integral equation (5.7) one can show:

$$(1-\mathbf{K})(1+\beta\partial_\beta)\varepsilon = \omega \,, \tag{5.14}$$

where $\mathbf{K}$ is the kernel

$$K(\mathbf{k},\mathbf{k}') = G(\mathbf{k},\mathbf{k}') f_{\mp}(\mathbf{k}') \,. \tag{5.15}$$

Similarly

$$(1-\mathbf{K})\partial_\mu\varepsilon = -1 \,. \tag{5.16}$$

Using now $(1-\mathbf{K})^{-1} = \sum_{m=0}^{\infty} \mathbf{K}^m$ one obtains

$$\mathcal{E} = \sum_{m=0}^{\infty} \int \frac{d\mathbf{k}}{(2\pi)^d} \; f_{\mp}(\mathbf{k}) \; \mathbf{K}^m * \omega \tag{5.17}$$

$$n = \sum_{m=0}^{\infty} \int \frac{d\mathbf{k}}{(2\pi)^d} \; f_{\mp}(\mathbf{k}) \; \mathbf{K}^m * 1 \,. \tag{5.18}$$

So far, our equations are quite general and valid whether the underlying theory is Lorentz invariant or not. For the remainder of this article, we assume the theory is Lorentz invariant with:

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} \,. \tag{5.19}$$

We can now derive an expression we will need for the pressure. Introduce the 4-vectors $k_\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ with $k^\mu k_\mu = m^2$. The kernel $\mathbf{G}$ can only be a function of the Mandelstam variable:

$$s(\mathbf{k},\mathbf{k}') \equiv (k_\mu + k'_\mu)^2 = 2(m^2 + k^\mu k'_\mu) \,. \tag{5.20}$$

(The other possible variable $t = (k_\mu - k'_\mu)^2 = 4m^2 - s$ is not independent.) We will need

$$\nabla_{\mathbf{k}}\, s = \frac{2}{\omega_{\mathbf{k}}}(\omega_{\mathbf{k}'}\mathbf{k} - \omega_{\mathbf{k}}\mathbf{k}') \,, \qquad \nabla_{\mathbf{k}'}\, s = \frac{2}{\omega_{\mathbf{k}'}}(\omega_{\mathbf{k}}\mathbf{k}' - \omega_{\mathbf{k}'}\mathbf{k}) \,. \tag{5.21}$$

The above implies for the kernel $\mathbf{G}$:

$$\omega_{\mathbf{k}}\, \nabla_{\mathbf{k}}\, G(\mathbf{k},\mathbf{k}') = -\omega_{\mathbf{k}'}\, \nabla_{\mathbf{k}'}\, G(\mathbf{k},\mathbf{k}')\,. \tag{5.22}$$

Using the integral equation for ε one can now show

$$(1-\widetilde{\mathbf{K}})\;\nabla_{\mathbf{k}}\,\varepsilon = \nabla_{\mathbf{k}}\,\omega\,, \tag{5.23}$$

where $\widetilde{K}(\mathbf{k},\mathbf{k}') = \omega_{\mathbf{k}}{}^{-1}K(\mathbf{k},\mathbf{k}')\omega_{\mathbf{k}'}$. This leads to

$$p = \frac{1}{d}\sum_{m=0}^{\infty}\int \frac{d\mathbf{k}}{(2\pi)^d}\; f_{\mp}(\mathbf{k})\;\mathbf{k}\cdot\widetilde{\mathbf{K}}^m * \nabla\omega\,. \tag{5.24}$$

What remains is to determine the kernel $\mathbf{G}$. Consider the one-point function $\langle T^{\mu}_{\mu}\rangle_\beta$. From first principles one has ($\mu=0$):

$$\langle T^{\mu}_{\mu}\rangle_\beta = \frac{1}{Z}\,\mathrm{Tr}\left(T^{\mu}_{\mu}e^{-\beta H}\right). \tag{5.25}$$

The left hand side can be computed from the free energy eq. (2.6) whereas the right hand side can be computed by inserting a multi-particle resolution of the identity into the trace and expressing the result in terms of the form factors of T^{μ}_{μ}. In this way we will relate the kernel to a certain *zero temperature* form factor. This computation was carried out for 2D integrable theories in [32], and we will subsequently follow this reference closely.

Let us first determine $\langle T^{\mu}_{\mu}\rangle$ from the free energy. Since $\langle T^{\mu}_{\mu}\rangle_\beta = \mathcal{E} + (D-1)\,p$, using eqs. (5.17), (5.24) one obtains:

$$\langle T^{\mu}_{\mu}\rangle_\beta = \sum_{m=0}^{\infty}\int \frac{d\mathbf{k}}{(2\pi)^d}\; f_{\mp}(\mathbf{k})\;\left(\mathbf{K}^m * \omega - \mathbf{k}\cdot\widetilde{\mathbf{K}}^m * \nabla\omega\right). \tag{5.26}$$

Next we describe a form-factor computation of $\langle T^{\mu}_{\mu}\rangle_\beta$. Introduce particle states $|\mathbf{k}\rangle$ normalized as follows:

$$\langle \mathbf{k}'|\mathbf{k}\rangle = (2\pi)^d\omega_{\mathbf{k}}\;\delta^{(d)}(\mathbf{k}-\mathbf{k}')\,. \tag{5.27}$$

Inserting a multi-particle resolution of the identity in eq. (5.25) one finds

$$\langle T^{\mu}_{\mu}\rangle_\beta = \frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\underline{\underline{d\mathbf{k}_i}}\,e^{-\beta\omega_{\mathbf{k}_i}}\right]\langle \mathbf{k}_n\cdots\mathbf{k}_1|T^{\mu}_{\mu}|\mathbf{k}_1\cdots\mathbf{k}_n\rangle\,. \tag{5.28}$$

The form factors $\langle \mathbf{k}_n\cdots|T^{\mu}_{\mu}|\mathbf{k}_1\cdots\rangle$ are the zero temperature form factors. These form factors have two kinds of contributions: disconnected contributions proportional to δ functions and the connected part which involves no δ functions. As explained in [31, 32], the effect of the disconnected parts is two-fold: some cancel the overall $1/Z$ and others convert the $e^{-\beta\omega}$ into the filling fractions $f_{\mp}$. Because the partition function involves the quasi-particle energy ε, the filling fractions f are also the quasi-particle energy ones given in eq. (5.13). The final result is then:

$$\langle T^{\mu}_{\mu}\rangle_\beta = \sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\underline{\underline{d\mathbf{k}_i}}\;f_{\mp}(\mathbf{k}_i)\right]\langle \mathbf{k}_n\cdots\mathbf{k}_1|T^{\mu}_{\mu}|\mathbf{k}_1\cdots\mathbf{k}_n\rangle_{\mathrm{conn}}\,. \tag{5.29}$$

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Finally comparing eq. (5.29) with the n-th term in the sum eq. (5.26) one finds

$$\langle \mathbf{k}_n \cdots \mathbf{k}_1 | T^\mu_\mu | \mathbf{k}_1 \cdots \mathbf{k}_n \rangle_{\rm conn} = (\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_n} - \mathbf{k}_1 \cdot \mathbf{k}_n) G(\mathbf{k}_n, \mathbf{k}_{n-1}) \cdots G(\mathbf{k}_3, \mathbf{k}_2) G(\mathbf{k}_2, \mathbf{k}_1) + \text{perm.} \tag{5.30}$$

The above factorization is exact for the 2D integrable theories as a consequence of factorizability of the S-matrix; for non-integrable theories it is likely to be an approximation. The above equation implies:

$$\langle \mathbf{k} | T^\mu_\mu | \mathbf{k} \rangle_{\rm conn} = m^2 \tag{5.31}$$

which is a normalization condition. The kernel is then identified with a two-particle to two-particle form factor:

$$G(\mathbf{k}, \mathbf{k}') = \frac{1}{2k^\nu k'_\nu} \; \langle \mathbf{k}, \mathbf{k}' | T^\mu_\mu | \mathbf{k}', \mathbf{k} \rangle_{\rm conn} \,, \tag{5.32}$$

where $2k^\mu k'_\mu = s - 2m^2$.

In the sequel we will need a scaling equation obeyed by the kernel. $\mathbf{G}$ depends on the momenta $\mathbf{k}$ and couplings and masses denoted as λ with scaling dimension d_λ. Using only the fact that $\mathbf{G}$ has scaling dimension $(1-d)$ one has:

$$G(e^l \mathbf{k}, e^l \mathbf{k}'; e^{d_\lambda l} \lambda) = e^{(1-d)l} \; G(\mathbf{k}, \mathbf{k}'; \lambda) \,. \tag{5.33}$$

It is straightforward to extend the above formulas to many particles. Let there be N particle species, each with statistical parameter $s_a = \pm 1$, where $s_a = 1$ corresponds to a boson, and m_a their masses, $a = 1, \ldots, N$. The equations become:

$$\log Z = -V \sum_a \int \frac{d\mathbf{k}}{(2\pi)^d} \; s_a \log\left(1 - s_a e^{-\beta\varepsilon_a}\right) \tag{5.34}$$

$$\varepsilon_a = (\omega_{\mathbf{k},a} - \mu_a) + \frac{1}{\beta} \sum_b \mathbf{G}_{ab} * s_b \log\left(1 - s_b e^{-\beta\varepsilon_b}\right) , \tag{5.35}$$

where $\omega_{\mathbf{k},a}$ depends on m_a. The kernels are

$$G_{ab}(\mathbf{k}, \mathbf{k}') = \frac{1}{2k^\nu_a k'_{\nu,b}} \langle \mathbf{k}, a; \mathbf{k}', b | T^\mu_\mu | \mathbf{k}', b; \mathbf{k}, a \rangle \,. \tag{5.36}$$

In summary, the quasi-particle approach we have developed is implemented as follows. One first computes the kernel $\mathbf{G}$ from the zero temperature form factor eq. (5.32) to whatever order in perturbation theory is feasible. One can then solve for the quasi-particle energy ε by numerically solving the integral equation (5.7). Finally the partition function is given by the simple eq. (5.9).

6. Comparison with perturbation theory in the Matsubara formalism

The standard methodology of finite temperature field theory involves perturbation theory where one must sum over Matsubara frequencies. Since this looks completely different from the approach of the last section, it is important to check that both approaches agree at least to lowest order. In this section we perform this check for the ϕ^4 scalar field theory in any dimension. The lowest order contribution to the free energy, eq. (6.17) below, is well-known [1, 2]. In the latter formalism, this lowest order correction is a 2-loop vacuum Feynman diagram. As we will see, to obtain this standard Matsubara result in our formalism is somewhat subtle.

The interacting scalar theory is defined by the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{6.1}$$

The usual stress-energy tensor is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu}\mathcal{L} . \tag{6.2}$$

The problem with the usual stress-energy tensor, is that even in the conformal limit where the theory is free and massless ($\lambda, m = 0$), the above stress-energy tensor is not traceless. It is known in this limit how the stress-tensor can be improved so that it is still conserved but now traceless [33]:

$$T^{\rm new}_{\mu\nu} = T_{\mu\nu} + \frac{(d-1)}{2d}\left(\eta_{\mu\nu}\partial^\alpha(\phi\partial_\alpha\phi) - \partial_\mu(\phi\partial_\nu\phi)\right) . \tag{6.3}$$

In the interacting case, the further improvement eq. (3.17) is necessary:

$$T^\mu_\mu = \frac{\dot\lambda}{4!}\phi^4 , \tag{6.4}$$

where $\dot\lambda$ is the beta function for flow toward the infra-red. ($\dot\lambda$ differs by a sign from the usual convention of flow toward the ultra-violet in high energy physics.) Since the field ϕ has mass dimension $(D-2)/2$ in D dimensions and the operator ϕ^4 has dimension $2(D-2)$,

$$\dot\lambda = (4-D)\lambda + O(\lambda^2) . \tag{6.5}$$

We can now compute the form factor that determines the kernel. Expand the field in the usual way:

$$\phi(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) , \tag{6.6}$$

where

$$[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}] = (2\pi)^d \ \delta^{(d)}(\mathbf{k}-\mathbf{k}') . \tag{6.7}$$

The states $|\mathbf{k}\rangle$ with the normalization in section 9 are then:

$$|\mathbf{k}\rangle = \sqrt{\omega_{\mathbf{k}}}\ a^\dagger_{\mathbf{k}}|0\rangle . \tag{6.8}$$

Expanding the fields ϕ in eq. (6.4) and evaluating the matrix element one finds:

$$\langle \mathbf{k}, \mathbf{k}'|T^\mu_\mu|\mathbf{k}', \mathbf{k}\rangle = \frac{\dot\lambda}{4} . \tag{6.9}$$

Thus the kernel $\mathbf{G}$ to lowest order is

$$G(\mathbf{k},\mathbf{k}') = \frac{\dot{\lambda}}{4(s-2m^2)}\,, \tag{6.10}$$

where $s(\mathbf{k},\mathbf{k}')$ is the Mandelstam variable eq. (5.20).

The integral gap equation implies that the lowest order correction to ε is:

$$\varepsilon \approx \omega_{\mathbf{k}} + \frac{1}{\beta} G * \log(1-e^{-\beta\omega})\,. \tag{6.11}$$

Inserting this into the free energy, one finds

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 + O(\lambda^2)\,, \tag{6.12}$$

where

$$\mathcal{F}_1 = \frac{1}{\beta}\int \underline{\underline{d\mathbf{k}}} \int \underline{\underline{d\mathbf{k}'}}\; f_0(\mathbf{k})\,\omega_{\mathbf{k}}\, G(\mathbf{k},\mathbf{k}')\, \log(1-e^{-\beta\omega_{\mathbf{k}'}})\,. \tag{6.13}$$

Above, the filling fraction is the free-field bosonic one:

$$f_0(\mathbf{k}) = \frac{1}{e^{\beta\omega_{\mathbf{k}}}-1} \tag{6.14}$$

Next we express the kernel as a derivative. For simplicity we only consider the massless case. One can easily show:

$$\frac{\omega_{\mathbf{k}'}}{\omega_{\mathbf{k}}}\, \mathbf{k}\cdot\nabla_{\mathbf{k}'}\left(\frac{1}{s}\right) = \frac{1}{s}\,, \qquad -\mathbf{k}'\cdot\nabla_{\mathbf{k}'}\left(\frac{1}{s}\right) = \frac{1}{s}\,. \tag{6.15}$$

Using these two equations in turn, and integrating by parts, one has:

$$\begin{aligned} \mathcal{F}_1 &= -\frac{\dot{\lambda}}{4}\int \underline{\underline{d\mathbf{k}}} \int \underline{\underline{d\mathbf{k}'}}\; f_0(\mathbf{k}) f_0(\mathbf{k}')\, \frac{\mathbf{k}\cdot\mathbf{k}'}{s} \\ &= (D-2)\mathcal{F}_1 + \frac{\dot{\lambda}}{4}\int \underline{\underline{d\mathbf{k}}} \int \underline{\underline{d\mathbf{k}'}}\; f_0(\mathbf{k}) f_0(\mathbf{k}')\, \frac{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{s}\,. \end{aligned} \tag{6.16}$$

The extra $(D-2)\mathcal{F}_1$ comes from $\nabla_{\mathbf{k}}\cdot(\mathbf{k}/k) = (D-2)/k$. Adding the two above equations one obtains

$$\mathcal{F}_1 = \frac{\lambda}{8}\left(\int \underline{\underline{d\mathbf{k}}} f_0(\mathbf{k})\right)^2\,. \tag{6.17}$$

The above equation is the well-known lowest order correction to the free energy [1, 2]. Note that in $4D$ one needs to keep the leading term $(4-D)\lambda$ in the beta function (6.5) through to the last step in order to obtain eq. (6.17).

7. Thermal gaps and $c_{\rm qs}$ for interacting fixed points

In this section we analytically study the integral gap equation for relativistic theories near an RG fixed point. In this way we will obtain expressions for $c_{\rm qs}$ for an interacting fixed point. Throughout this section the chemical potential $\mu = 0$. For simplicity we first treat a theory with a single particle.

In the approach developed in section 5, $C_{\rm qs}$ is given by the formula:

$$C_{\rm qs}(\beta) = \mp\frac{\beta^d}{a_d}\int\frac{d\mathbf{k}}{(2\pi)^d}\ \log\left(1\mp e^{-\beta\varepsilon}\right). \tag{7.1}$$

As we argued in section 3, at a fixed point $C_{\rm qs}$ is a constant independent of β. At a fixed point the theory is expected to be massless. This can occur in the ultra-violet (UV), i.e. at high temperature or high energy compared to the mass where $\omega_{\mathbf{k}} \approx \sqrt{\mathbf{k}^2} \equiv k$. We will focus on such UV fixed points, but some of our analysis can also apply to infra-red fixed points at low temperature since such a theory is expected to be massless from the beginning for all $\mathbf{k}$ and again $\omega_{\mathbf{k}} = k$. The fixed point properties are then determined by the solution to the massless integral gap equation:

$$\varepsilon = k\ \pm\ \frac{1}{\beta}G * \log\left(1\mp e^{-\beta\varepsilon}\right). \tag{7.2}$$

At a fixed point, $\mathbf{G}$ should depend only on dimensionless parameters λ with $d_\lambda = 0$, so that eq. (5.33) implies

$$G(e^l\mathbf{k}, e^l\mathbf{k}') = e^{(1-d)l}G(\mathbf{k},\mathbf{k}'). \tag{7.3}$$

Then from eq. (7.2) one can show

$$\varepsilon(e^l\mathbf{k},\beta) = e^l\varepsilon(k, e^l\beta) \qquad \text{(at a fixed point)}. \tag{7.4}$$

The fixed point value of $C_{\rm qs}$ is then:

$$C_{\rm qs}(\beta) = \mp\frac{\beta^d}{a_d}\int \underline{\underline{d\mathbf{k}}}\ k\ \log\left(1\mp e^{-\beta\varepsilon}\right), \tag{7.5}$$

where now ε satisfies the massless integral gap equation (7.2). By rescaling $\mathbf{k}$ in the above equation one shows that $C_{\rm qs}(e^l\beta) = C_{\rm qs}(\beta)$. In summary, if ε satisfies the massless integral gap equation (7.2), and $\mathbf{G}$ the scaling equation (7.3) then $C_{\rm qs}$ in eq. (7.1) is a constant $c_{\rm qs}$ independent of β.

An important feature of the massless integral gap equation (7.2) is that ε generally develops a thermal gap $\Delta \propto T$ at high temperatures. When k is small compared to the temperature T, the k-term on the r.h.s. of (7.2) can be dropped and $\beta\varepsilon$ is approximately constant. Let us define the dimensionless thermal gap parameter g so that:

$$\varepsilon \approx gT \equiv \Delta, \qquad k \ll T, \qquad T\ \text{large}. \tag{7.6}$$

For k much higher than T then $\varepsilon \approx k$. These features are shown in figure 1. In the limit of $T \to \infty$, $\varepsilon = \Delta$ for all k, and g is a solution to the *algebraic gap equation*:

$$g = \pm h\log(1\mp e^{-g}), \tag{7.7}$$

where h is the dimensionless parameter:

$$h = \int \underline{\underline{d\mathbf{k}}}\ G(0,\mathbf{k}). \tag{7.8}$$

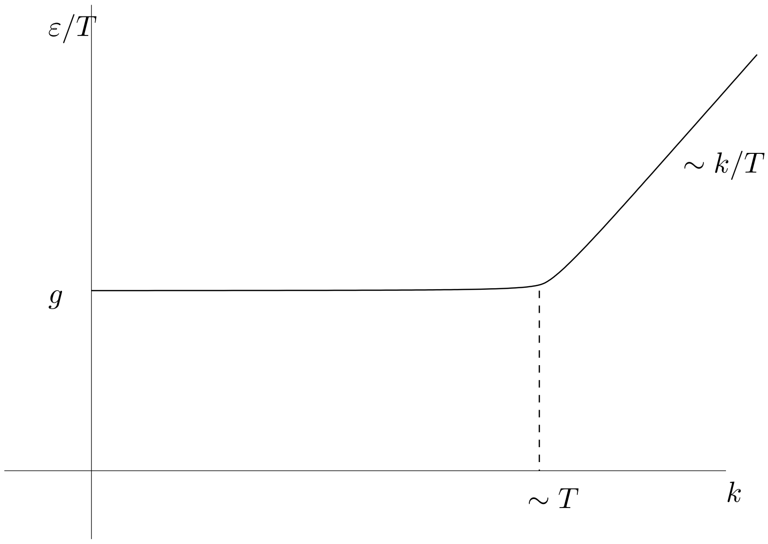

Figure 1: The thermal gap.

Henceforth, by gap equation we will mean the algebraic equation (7.7) rather than the integral gap equation (5.7). The interactions are encoded in the parameter h which we will refer to as *interaction parameters*.

The quantity $c_{\rm qs}$ should only depend on the gap parameter g. It can be computed to a very good approximation, if not exactly, as follows. In eq. (7.1) we wish to trade the integral over $\mathbf{k}$ for an integral over ε. Differentiating eq. (7.2) with respect to k one finds

$$k = k\partial_k \varepsilon \;\mp\; \frac{1}{\beta} k\partial_k G * L\,, \tag{7.9}$$

where we have defined $L = \log(1 \mp e^{-\beta\varepsilon})$. Inserting this into eq. (7.1):

$$c_{\rm qs} = \mp \frac{\beta^d}{a_d} \int \underline{\underline{d\mathbf{k}}} \left(k\partial_k \varepsilon \mp \frac{1}{\beta} k\partial_k G * L \right) L(\mathbf{k})\,. \tag{7.10}$$

Using the property eq. (5.22) and a few integrations by parts one obtains:

$$c_{\rm qs} \propto \int_\Delta^\infty d\varepsilon\; k^{d-1} \left(\pm L(\varepsilon) + \frac{\beta\varepsilon}{e^{\beta\varepsilon} \mp 1} \right). \tag{7.11}$$

When $\varepsilon > \Delta$, $k \approx \varepsilon$, so we approximate k by ε in the above equation. The final result is:

$$c_{\rm qs}(g) = \frac{1}{\Gamma(D-1)\zeta(D)}\; \mathcal{L}_D(g)\,, \tag{7.12}$$

where

$$\mathcal{L}_{d+1}(g) \equiv \frac{1}{(d+1)} \int_g^\infty dx \left(\mp x^{d-1} \log(1 \mp e^{-x}) + \frac{x^d}{e^x \mp 1} \right). \tag{7.13}$$

(We have used $\int d\mathbf{k} = \Omega_d \int k^{d-1} dk$, $\Omega_d = (2\pi^{d/2}/\Gamma(d/2))$, and the duplication formula for the Γ-function.)

The above integral can be expressed in terms of the the polylogarithm, or Jonquière, function:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \,. \tag{7.14}$$

For $D = 2, 3, 4$ one finds:

$$\begin{aligned}
\mathcal{L}_2(g) &= \pm\frac{g}{2}\mathrm{Li}_1(\pm e^{-g}) \pm \mathrm{Li}_2(\pm e^{-g}) \\
\mathcal{L}_3(g) &= \pm\frac{g^2}{3}\mathrm{Li}_1(\pm e^{-g}) \pm g\mathrm{Li}_2(\pm e^{-g}) \pm \mathrm{Li}_3(\pm e^{-g}) \\
\mathcal{L}_4(g) &= \pm\frac{g^3}{4}\mathrm{Li}_1(\pm e^{-g}) \pm g^2\mathrm{Li}_2(\pm e^{-g}) \pm 2g\mathrm{Li}_3(\pm e^{-g}) \pm 2\mathrm{Li}_4(\pm e^{-g}) \,.
\end{aligned} \tag{7.15}$$

Note that $\mathrm{Li}_1(z) = -\log(1-z)$.

Since the argument of the polylogarithms is always $\pm e^{-g}$, it is convenient to define the gap variable:

$$z \equiv e^{-g} \,. \tag{7.16}$$

To simplify further the subsequent notation, introduce the functions:

$$\mathrm{Lr}_n(z) = \mathrm{Li}_n(z) + \sum_{r=1}^{n-2} \frac{(-)^r}{r!} \log^r |z| \mathrm{Li}_{n-r}(z) + (-)^n \frac{(n-1)}{n!} \log^{n-1} |z| \log(1-z) \,. \tag{7.17}$$

Comparing with eq. (7.15), one finds

$$\mathcal{L}_{d+1}(g) = \pm\Gamma(d)\, \mathrm{Lr}_{d+1}(\pm e^{-g}) \,. \tag{7.18}$$

One then has the simple formula for real z:

$$c_{\mathrm{qs}}(z) = \frac{1}{\zeta(d+1)}\; s\mathrm{Lr}_{d+1}(sz) \equiv c_D(z, s) \,, \tag{7.19}$$

where as before, $s = +/-$ corresponds to bosons/fermions.

For many particles the gap equation becomes

$$g_a = \sum_b h_{ab}\; s_b \log(1 - s_b e^{-g_b}) \,, \tag{7.20}$$

where

$$h_{ab} = \int \underline{\underline{d\mathbf{k}}}\; G_{ab}(0, \mathbf{k}) \,. \tag{7.21}$$

In terms of the z_a,

$$z_a = \prod_b (1 - s_b z_b)^{-h_{ab} s_b} \,. \tag{7.22}$$

The quantity c_{qs} then becomes a sum:

$$c_{\mathrm{qs}} = \sum_a \; c_D(z_a, s_a) \tag{7.23}$$

Let us summarize the results of this section. The field theory has been deconstructed into the interaction parameters h_{ab}, which determine the thermal gap parameters z_a through the gap equation. The fixed point value of the central charge $c_{\rm qs}$ is then a function of the z's.

Since h_{ab} determine the fixed point properties, they must be an RG invariant functions of the couplings. For the 2D integrable theories, it is well known how the h's arise from coupling constants in the exact S-matrix. (See section 10.2.) Since we have relaxed the integrability, let us illustrate how to compute h to lowest order in the ϕ^4 theory in 2D, which is not integrable. The kernel $\mathbf{G}$ was determined to lowest order in section 6. The interaction parameter h to lowest order is then:

$$h = \frac{\dot{\lambda}}{8m} \int \frac{d\mathbf{k}}{2\pi} \frac{1}{\omega_{\mathbf{k}}^2} \, . \tag{7.24}$$

The above integral is convergent, and using $\dot{\lambda} = 2\lambda + \cdots$, to lowest order one finds

$$h = \frac{\lambda}{8m^2} \, . \tag{7.25}$$

Since λ has dimension 2, the ratio λ/m^2 is a dimensionless coupling constant, and thus so is h. The ϕ^4 model may be considered as the weak coupling limit ($b \to 0$) limit of the sinh-Gordon model defined in eq. (10.13) where $h \propto b^2$. However the integrable sinh-Gordon model actually has $h = 1$ indepependent of b (see below) which shows the importance of the higher order corrections.

For higher dimensions it is not clear how the structure of the kernel $\mathbf{G}$ can lead to finite RG invariant h's, and this will remain an important unanswered question in this paper.

8. Classical statistical mechanics: Ising-like models

In this section we consider the application of the above formalism to classical statistical mechanics in D spacial dimensions. Since the physical context here is very different, the subscript ${}_{\rm cl}$ will be used to denote quantities in classical statistical mechanics to avoid the likely confusion. The meaning of $c_{\rm qs}$ in this context is the following. Consider a statistical mechanical system on a D dimensional lattice where all dimensions of the lattice are infinite except for one of length L, i.e. the volume of the lattice is VL where V is a $D-1$-dimensional volume and $V \to \infty$. At the critical classical temperature T_c, the partition function should only depend on L. If periodic boundary conditions are imposed in the finite size direction, then L is the same as a finite inverse temperature β. Thus the results of section 3 imply:

$$\lim_{V \to \infty} \frac{1}{V} \log Z_{\rm cl} = c_{\rm qs} \, a_{D-1} \, L^{1-D} \, . \tag{8.1}$$

Consider an Ising-like model on a D dimensional square lattice. At each site i there is a spin degree of freedom $\sigma_i = \pm 1$. The partition function and classical energy E are

$$Z_{\rm cl} = \sum_{\rm config.} e^{-\beta_{\rm cl} E} \, , \qquad E \equiv J \sum_{<i,j>} \sigma_i \sigma_j \, , \tag{8.2}$$

where $\langle i, j \rangle$ denotes nearest neighbors, and $\beta_{\rm cl} = 1/T_{\rm cl}$. For $D = 2$ the usual Ising model is equivalent to a relativistic quantum field theory of a free Majorana fermion in euclidean $2D$. Though this is well-known, it is very non-trivial and relies on mappings to the XY spin chain and a Jordan-Wigner transformation.

No higher dimensional version of this is known. We can explore the possibility of a higher dimensional version using the gap equation. In a disordered phase, since the spins are disordered, one expects that $\langle E \rangle = 0$. From the first law of thermodynamics, one then obtains that the free energy is proportional to the entropy: $F_{\rm cl} = -S_{\rm cl}/\beta_{\rm cl}$. Now suppose that at the critical point $\log Z_{\rm cl}$ is equivalent to a field theory functional integral at an ultra-violet fixed point. Let us clarify that the temperature of the field theory T and the classical temperature $T_{\rm cl}$ have nothing to do with each other. As is well-known in the 2D case, the classical temperature appears as a coupling constant in the zero temperature field theory, the mass of the fermion being proportional to $T_{\rm cl} - T_c$ where T_c is the critical temperature. Using the results of section 7, one finds in the ultra-violet limit of the field theory for the entropy:

$$S \approx \pm V \int \frac{d\mathbf{k}}{(2\pi)^d} \, \log(1 \mp e^{-g}) \, , \tag{8.3}$$

where g is the thermal gap. Since $V/(2\pi)^d$ is approximately a level density, then $S \approx \pm \sum \log(1 \mp e^{-g})$. From this equation, one sees that $g = \log 2$ in the bosonic case is special since then the entropy resembles $S = \sum \log 2$, which suggests a model with 2 states per site, such as an Ising model.

Though the above discussion is very rough, $g = \log 2$ indeed leads to the same central charge $c_{\rm qs} = 1/2$ as for the 2D Ising model (7.12). For $3D$, the same formulas again give a rational result, $c_{\rm qs} = 7/8$. This rationality is quite remarkable given the complexity of the various functions involved. The existence of rational theories will be explored in the next section. In $4D$, rationality ceases and one finds $c_{\rm qs} = {\rm Lr}_4(1/2)/\zeta(4) = 0.97805427\ldots$. Clearly more work is needed in the 3D case, for example one could check whether for an infinite slab of thickness L, $\log Z_{\rm cl}/V$ scales as $7\zeta(3)/16\pi L^2$.

9. Rational theories

In 2D, theories with a rational $c_{\rm qs}$ are of particular importance. For instance unitary theories with $0 < c < 1$ have been classified and are the so-called minimal models with rational $c = 1 - 6/(n+2)(n+3)$, $n = 1, 2, \ldots$ [14, 39]. Also, all known integrable scattering theories lead to thermodynamic Bethe ansatz equations with a rational $c_{\rm qs}$ in the ultra-violet limit.

The function ${\rm Lr}_2(z)$ is the Roger's dilogarithm. Remarkably, the higher order functions ${\rm Lr}_n$ have also appeared in the mathematical literature [43]. They were not originally defined in terms of the integral of section 7. Rather, these functions arose as the unique functions that remove logarithmic terms in certain functional equations satisfied by ${\rm Li}_n(z)$. We can now formulate the following problem. A rational theory in D dimensions requires the

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following identity:

$$\sum_{a=1}^{N} s_a \,\mathrm{Lr}_D(s_a z_a) = \mathcal{R}\,\zeta(D)\,, \tag{9.1}$$

where $s_a = \pm 1$ corresponds to the statistics and z_a are numbers that characterize the thermal gaps g_a, $z_a = e^{-g_a}$. If $\mathcal{R}$ is rational, then $c_{\rm qs} = \mathcal{R}$ is rational. We will refer to a relation of the form eq. (9.1) as a Rational Statistical Polylogarithmic Evaluation (RSPE). Every RSPE may be viewed as a non-trivial result in number theory.

Of course, a relation of the form eq. (9.1) does not guarantee the existence of a field theory with this $c_{\rm qs}$ in the UV. However one can try to go as far as possible in reconstructing some basic data of a possible theory. Given the z_a from an RSPE, one can use the gap equation to determine the interaction parameters h_{ab}. There then remains the difficult problem of finding a consistent field theory leading to kernels $\mathbf{G}_{ab}$ that yield the h_{ab}. In 2D, if an integrable theory exists with $c_{\rm qs}$, the theory can often be entirely reconstructed, i.e all the scattering amplitudes can be deduced, due to the tight constraints of integrability. The encouraging fact is that, as we show below, we have been able to reconstruct the field theories for all known one-particle (N=1) RSPE's in 2D. We have also been able to reconstruct a 2-particle theory in $2D$.

The classification of fixed points of field theories is then closely related to the classification of RSPE's. An RSPE is a subclass of the more general polylogarithmic ladders constructed in [43]. The latter do not have the restriction that the statistical parameters s_a enter the relation as in (9.1), though some polylogarithmic ladders can be brought into the form of RSPE's using certain functional transformations (see below). The gap equation (7.22) is essentially what is called the cyclotomic equation in the construction of polylogarithmic ladders. The forerunners of dilogarithmic ladders (d=1) were first found by Euler and Landen in the 1700's [41, 42]. New dilogarithmic ladders were not found until the 1930's by Coxeter and Watson. In the 1980's it was realized by Lewin that there is a vast extension of these previous few results. Despite this progress, polylogarithmic ladders are still considered somewhat mysterious and there are no classification theorems. Some of them have only been verified to high precision numerically. Ladders have even been discovered to order $D = 17$ [46]. In the sequel we will present a number of RSPE's that were deduced from results in [43 − 45], although they do not all appear there in the form they are given here; the diligent reader can easily verify them numerically (as we did).

Let us first describe the only results known that are valid for all dimension D. Polylogarithmic ladders are constructed using functional relations satisfied by $\mathrm{Lr}_n(z)$, without doing any explicit integrals. For any n, $\mathrm{Lr}_n(1) = \zeta(n)$. This corresponds to a free boson in any dimension with $c_{\rm qs} = 1$. The functions $\mathrm{Lr}_n(z)$ satisfy a number of functional equations that are valid for any n. The duplication relation reads:

$$\mathrm{Lr}_n(z) + \mathrm{Lr}_n(-z) = 2^{1-n}\mathrm{Lr}_n(z^2)\,. \tag{9.2}$$

From this it follows that $-\mathrm{Lr}_{d+1}(-1) = (1 - 2^{-d})\zeta(d+1)$. This corresponds to a free fermion in any dimension with

$$c_{d+1}(1,-1) = 1 - \frac{1}{2^d}\,, \qquad \text{(free fermion)}\,. \tag{9.3}$$

For any n one also has the inversion relation:

$$\mathrm{Lr}_n(-z) + (-)^n \mathrm{Lr}_n\left(-\frac{1}{z}\right) = C_n\,, \tag{9.4}$$

where C_n is a constant. For n odd, $C_n = 0$, whereas for n even $C_n = -2(1-2^{1-n})\zeta(n)$. Using $\mathrm{Lr}_n(0) = 0$ in the inversion relation, one finds:

$$c_{d+1}(\infty, -1) = 2\left(1 - \frac{1}{2^d}\right) \qquad (d \text{ odd})\,. \tag{9.5}$$

The above relation is fermionic, and since $z = \infty$, it is an interacting theory with gap $g = -\infty$ and interaction parameter $h = 1$. As we explain in the next subsection, in $2D$ this theory is realized in e.g. the sinh-Gordon model with $c_{\rm qs} = 1$. In higher dimensions with d odd, $c_{\rm qs} = 2(1-2^{-d})$ which is twice that of a free fermion and always more than a boson. We will refer to this case as the extreme-fermion since it possesses the highest possible $c_{\rm qs}$ for a fermionic theory.

Notice that the above results were obtained without doing any integrals explicitly, but rather just by using functional relations satisfied by the $\mathrm{Lr}_n(z)$. More complex polylogarithmic ladders are constructed in the same spirit. Since the more recently discovered ladders are for $N > 1$, a classification of 1-particle theories is essentially within reach. In the next section we carry out this classification.

10. Rational theories in $2D$

10.1 One particle RSPE's

Let us illustrate how ladders are constructed in the simple case of $N = 1$ (1-particle) and $D = 2$. The construction relies on additional functional relations which are valid at second order:

$$\begin{aligned} \mathrm{Lr}_2(z)\ &+ \mathrm{Lr}_2(1-z) = \zeta(2) \\ \mathrm{Lr}_2(z)\ &+ \mathrm{Lr}_2\left(-\frac{z}{(1-z)}\right) = 0\,. \end{aligned} \tag{10.1}$$

Using the first relation above, one sees that $\mathrm{Lr}_2(1/2) = \zeta(2)/2$. This, and the free fermion and free boson cases, were the relations known to Euler. Landen found more evaluations as follows. If r is a root to the polynomial equation $z^2 + z = 1$, then the above functional equations become linear equations for $\mathrm{Lr}_2(z)$ with argument $z = r, -r, -1/r$ and r^2. Let us take $r = (\sqrt{5}-1)/2$, which is the inverse of the golden ratio. This is the complete list of known RSPE's for $D = 2$ and $N = 1$, and has been known for over 220 years:

$$\begin{aligned} \mathrm{Lr}_2(1) &= -\mathrm{Lr}_2(-\infty) = \zeta(2) \\ \mathrm{Lr}_2\left(\frac{1}{2}\right) &= -\mathrm{Lr}_2(-1) = \frac{1}{2}\zeta(2) \\ \mathrm{Lr}_2(r^2) &= -\mathrm{Lr}_2(-r) = \frac{2}{5}\zeta(2) \\ \mathrm{Lr}_2(r) &= -\mathrm{Lr}_2\left(-\frac{1}{r}\right) = \frac{3}{5}\zeta(2)\,. \end{aligned}$$

statistics	c_{qs}	g	h	$S(0)$	s_f	field theory
fermion	1	$-\infty$	1	-1	1	sinh-Gordon
boson	1	0	0	1	1	free boson
fermion	1/2	0	0	1	-1	free fermion
boson	1/2	$\log 2$	-1	1	1	
fermion	2/5	$-\log r$	-1	-1	1	Lee-Yang
boson	2/5	$-\log r^2$	-2	1	1	
fermion	3/5	$-\log 1/r$	1/2	1	-1	$\mathcal{M}_{3,5}$
boson	3/5	$-\log r$	$-1/2$	1	1	

Table 1: Rational 1-particle theories in 2D.

Though it is generally believed that the above list is exhaustive, we could not find a proof in the literature. We suggest that a proof might be concocted based on proving that the known integrable scattering theories in 2D with one particle are exhaustive and are in correspondence with the above list.

Assuming that the relations eq. (10.2) are indeed exhaustive, this leads to the classification of the possible theories shown in table 1. In the table are listed the thermal gaps g and the interaction parameter h which can be inferred from the gap equation (7.7). The last 3 columns will be explained below.

10.2 Integrable reconstruction in 2D

Let us specialize the results of section 5 to integrable relativistic theories in $2D$ [35]. Let $A_a(\theta)$, $a = 1, \ldots, N$ denote particle creation operators where θ is the rapidity parameterizing the one-particle relativistic dispersion relation:

$$E = m_a \cosh\theta\,, \qquad \mathbf{k} = m_a \sinh\theta\,. \tag{10.2}$$

In this rapidity space the formulas for the partition function and integral gap equation take the form:

$$\log Z \;=\; -V \sum_a \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_a \cosh\theta s_a \log(1 - s_a e^{-\beta\varepsilon_a(\theta)}) \tag{10.3}$$

$$\varepsilon_a(\theta) \;=\; (m_a \cosh\theta - \mu_a) + \frac{1}{\beta}\sum_b \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\, G_{ab}(\theta-\theta') s_b \log(1 - s_b e^{-\beta\varepsilon_b(\theta')})\,. \tag{10.4}$$

We will consider only theories where the scattering is diagonal. The factorizable S-matrix $S_{ab}(\theta)$ then enters the exchange relation as follows:

$$A_a(\theta)A_b(\theta') = S_{ab}(\theta-\theta')A_b(\theta')A_a(\theta)\,. \tag{10.5}$$

In the case of $2D$ an alternative derivation of the kernel is possible [34, 35]. Consider the quantum field theory on a space of 1 dimensional length $L(=V)$. Requiring that the multi-particle wave-function is periodic leads to a quantization condition on the momenta $\mathbf{k}$:

$$e^{iL\mathbf{k}_i} \prod_{j\neq i} S_{ij}(\theta_i - \theta_j) = 1\,. \tag{10.6}$$

Taking the logarithm and differentiating with respect to the rapidity leads to a relation between ρ_l and ρ as in eq. (5.1) with the kernel:

$$G_{ab}(\theta) = -i\frac{\partial}{\partial\theta}\log S_{ab}\,. \tag{10.7}$$

This derivation has its origins in the thermodynamic Bethe ansatz (TBA). In many cases one can construct explicitly the Bethe ansatz wave-functions and derive equations such as eq. (10.6). For this reason we will refer to the above 2D equations as TBA equations.

On the other hand, using the arguments of the last section, the kernel $\mathbf{G}$ can be determined from the form factors. It was shown in [32] that if one defines the connected form factor in the proper way, then the resulting kernel is the same as in eq. (10.7).

We now attempt to identify the entries of the above table with specific integrable quantum field theories. All of the fermionic theories are well-known and their conventional names are in the last column. However each fermionic theory has a bosonic counterpart, which can be traced to the second functional relation eq. (10.1). As explained in [35], the statistics of the lagrangian fields does not necessarily correspond to the statistics in the partition function. First of all, there is no spin-statistics theorem in 2D. The S-matrix satisfies the unitarity condition $S(\theta)S(-\theta)=1$. At $\theta=0$, $S(0)^2=1$ has two possibilities: $S(0)=\pm1$. The quantization condition eq. (10.6) imposes its own restrictions on the wave-functions. If $S(0)=-1$, two bosons are not allowed to have the same rapidity, and one should impose fermionic exclusion statistics. If $S(0)=1$, the situation is reversed. We can summarize this as follows. Let $s_f=\pm1$ denote the statistics of the fields, where $s_f=1$ is a boson. Let s denote the statistics of the partition function, as in eq. (10.3). Then $s=s_fS(0)$. Henceforth, in the 2D case, boson or fermion statistics will refer to the partition function statistics s.

The reason that half the theories in Table 1 have not been identified is that nobody has ever made sense of an interacting *bosonic* TBA. It has even been suggested that interactions always turn bosons into fermions in 2D [37]. This issue was addressed in [47], where a hypothetical model with S-matrix given by the well-known sinh-Gordon one, but with the wrong statistics, was investigated. It was found that $c_{\rm qs}$ was ill-defined in the UV, which shows one cannot simply change the statistics of known theories at will. On the other hand, as table 1 shows, it is possible to have well-defined $c_{\rm qs}$ for a bosonic TBA.

For the remainer of this section we describe how to reconstruct S-matrices from the data in Table 1. We first review a few basic ingredients of diagonal factorizable S-matrices (see for instance [36−38]). Unitarity and crossing symmetry require

$$S_{ab}(\theta)S_{ba}(-\theta)=1\,, \qquad S_{a\overline{b}}(\theta)=S_{ab}(i\pi-\theta)\,, \tag{10.8}$$

where $\overline{a}$ is the charge conjugate to a. S-matrices satisfying the above equations are products of the fundamental building blocks:

$$f_\alpha(\theta)=\frac{\sinh\frac12(\theta+i\pi\alpha)}{\sinh\frac12(\theta-i\pi\alpha)}\,. \tag{10.9}$$

If a particle is its own anti-particle, then crossing symmetry implies it is a product of the functions

$$F_\alpha(\theta) = f_\alpha(\theta) f_\alpha(i\pi - \theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)} . \tag{10.10}$$

The poles in the S-matrix are also severely constrained. If a simple pole exists in S_{ab} at $\theta = iu$, on the physical strip $0 < iu < i\pi$, and with positive imaginary residue, then this corresponds to a stable bound state, with mass

$$m^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u . \tag{10.11}$$

If this is a new particle, one must proceed to close the bootstrap. A negative residue pole is either the same particle in the crossed channel, or a sign of non-unitarity. We will need that f_α has a simple pole at $\theta = i\pi\alpha$ with residue $2i \sin \pi\alpha$. From this it follows that F_α has a simple pole at $i\pi\alpha$ with residue $2i \tan \pi\alpha$ and a simple pole at $i(1-\alpha)\pi$ with residue $-2i \tan \pi\alpha$.

The h values that appear in the gap equation for the 1d integrable theories are:

$$h_{ab} = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \frac{\partial \log S_{ab}(\theta)}{\partial\theta} . \tag{10.12}$$

Using this one easily can compute that for each factor of f_α, $h = \alpha - \mathrm{sign}(\alpha)$. For each F_α, $h = -\mathrm{sign}(\alpha)$.

All the theories with $h \neq 0$ are interacting. Let us first go over the identification of the interacting fermionic theories, all of which are known.

Fermionic theory with $c_{\mathrm{qs}} = 1$. Here since $h = 1$ the S-matrix must be $F_{-\alpha}$ with α positive. For α positive, the pole on the physical strip has negative residue and thus does not correspond to a new particle, so the bootstrap is already closed. This is a one-parameter family of theories corresponding to the sinh-Gordon model, with lagrangian:

$$S_{\mathrm{shG}} = \int d^2x \, \left(\frac{1}{2}(\partial\phi)^2 + \Lambda \cosh(b\sqrt{4\pi}\phi) \right) . \tag{10.13}$$

The S-matrix is known to be $F_{-\alpha}$ where $\alpha = b^2/(2 + b^2)$. ϕ is a bosonic field, so $s_f = 1$, but since $S(0) = -1$, the partition function is fermionic.

Fermionic theory with $c_{\mathrm{qs}} = 2/5$. Since here $h = -1$, the S-matrix must be F_α with α positive. However for arbitrary $\alpha > 0$, the physical pole will correspond to a new particle and this is no longer a one-particle theory. The only solution is to take $\alpha = 2/3$. Then, since $\cos 2\pi/3 = -1/2$, the pole corresponds to a particle of the same mass as the original particle and can thus be identified with it. Thus the bootstrap closes with one particle. This model is known as the Lee-Yang model [48]. Generally, $c_{\mathrm{qs}} = c - 24\Delta_{\mathrm{min}}$ where c is the Virasoro central charge and Δ_{min} is the lowest conformal dimension of the fields. Here, $c = -22/5$ and $\Delta_{\mathrm{min}} = -1/5$, giving $c_{\mathrm{qs}} = 2/5$. The residue of the pole is negative, which is an indication of the non-unitarity of the theory.

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Fermionic theory with $c_{\text{qs}} = 3/5$. Here since $h = 1/2$ the S-matrix must be $f_{-1/2} = -i \tanh \frac{1}{2}(\theta - i\pi/2)$. Though f_α for general alpha cannot correspond to a 1-particle theory since it is not crossing symmetric, $\alpha = -1/2$ is special since it is at least anti-crossing symmetric: $f_{-1/2}(i\pi-\theta) = -f_{-1/2}(\theta)$. This model is known [49] to be the Φ_{13} perturbation of the $\mathcal{M}_{3,5}$ minimal model, with $c = -3/5$ and $\Delta_{\text{min}} = -1/20$, giving $c_{\text{qs}} = 3/5$.

We now turn to the bosonic theories. Note that they all occur at the same central charge as the fermionic ones. This does not imply they are the same theory, but only that they are the same at the UV fixed point. A theory is defined by its fixed point in the UV and by the operator which perturbs it away from the fixed point [50]. For the last 4 entries of table 1, we only indicate the UV fixed point theory.

Bosonic theory with $c_{\text{qs}} = 1/2$. This is the most interesting case since the UV central charge is that of the Ising model, of which a great deal is already known. Let us attempt to reconstruct a consistent S-matrix for this theory deferring its interpretation. Since $h = -1$, the S-matrix must be F_α with α positive. For the same reasons given above for Lee-Yang, α must be $2/3$ otherwise the bootstrap doesn't close. We propose the following S-matrix:

$$S(\theta) = pF_{2/3}(\theta)\,, \qquad p = \pm 1\,. \tag{10.14}$$

We now carefully investigate the consistency of this theory. For $p = -1$, the residue of the pole at $\theta = 2\pi i/3$ is positive, which is consistent with the theory being unitary. However the S-matrix now satisfies the bootstrap equation with an extra minus sign:

$$S(\theta) = pS\left(\theta + i\frac{\pi}{3}\right) S\left(\theta - i\frac{\pi}{3}\right). \tag{10.15}$$

This minus sign can be removed by $S \to -S$, however this spoils the unitarity of the theory. For these reasons, the above S-matrix is not entirely consistent and its physical interpretation will remain an open question. The same is true for the next two bosonic theories.

Bosonic theory with $c_{\text{qs}} = 2/5$. Here $h = -2$ which implies the S-matrix must be $F_\alpha F_{\alpha'}$. As in previous cases, the bootstrap will not close unless $\alpha = \alpha' = 2/3$:

$$S(\theta) = F_{2/3}(\theta)^2 \tag{10.16}$$

The resulting S-matrix has only double poles for the physical particle. Since $S(0) = 1$, this model has a single bosonic field. Further analysis is needed to determine whether it is equivalent to the Lee-Yang model, or a different off-critical perturbation.

Bosonic theory with $c_{\text{qs}} = 3/5$. Since $h = -1/2$, the S-matrix must be:

$$S(\theta) = -f_{1/2}(\theta)\,. \tag{10.17}$$

It has a pole at $i\pi/2$, with negative residue, indicating non-unitarity. Here, $s_f = 1$. As for the fermionic version, this is an integrable perturbation of the $\mathcal{M}_{3,5}$ non-unitary minimal model. Since the S-matrices are very similar in this case for the bosonic and fermionic theories ($f_{1/2}(\theta) = f_{-1/2}(-\theta)$), the two models are probably equivalent; again more analysis is needed.

10.3 A supersymmetric 2 particle theory in $2D$

As was stated before, all of the newly discovered ladders are multi-particle. Not all of them can be brought into the RSPE form. It is beyond our scope to do a comprehensive study of the vast number of results in [43], so we concentrate on a simple example that doesn't appear to correspond to anything already known. Using the duplication and inversion relations on one of the results in [43], one can show

$$c_2(y,1)+c_2(y,-1)=\frac{3}{4}\,,\qquad y=\sqrt{2}-1\,. \tag{10.18}$$

This is an interesting relation since, unlike the known TBA's that are purely fermionic, this is a mixed theory, with one boson and one fermion, suggesting a supersymmetric theory.

We can reconstruct the theory just from the data in the equation (10.18). It's a two particle theory with $z_1=z_2=\sqrt{2}-1$, but with $s_1=1, s_2=-1$. From the gap equation we determine the h's to be: $h_{11}=h_{22}=h_{12}=h_{21}=-1$. The choice of S-matrix:

$$S_{ab}(\theta)=-F_{2/3}(\theta)\,,\qquad \forall a,b=1,2 \tag{10.19}$$

yields these h values, and the bootstrap is closed for previously described reasons. The S-matrix shows no signs of non-unitarity. However the UV central charge is $c=3/4$ which is not in the minimal unitary series. Our interpretation is that the physically sensible theory is two copies of the above. This has $c=3/2$ with 4 particles and the supersymmetry is preserved. It is known that $c=3/2$ is the highest central charge of minimal unitary super-conformal series [51], supporting this idea.

11. RSPE's in $3D$ and $4D$

11.1 One particle theories in $3D$

As explained in the last section, the second identity in eq. (10.1) implies that in 2D a bosonic RSPE can be rewritten as a fermionic one with different gap parameter z. Because of bosonization, this is perhaps not surprising in 2D. In higher dimensions there is no analog of this identity so it is not possible in general to give both a bosonic and fermionic interpretation to a given RSPE.

The only known 1-particle RSPE's (due to Landen [42]) beyond the free boson and free fermion in $3D$ have again been known for over 220 years and are the following:

$$c_3\left(\frac{1}{2},1\right)=\frac{7}{8}\,,\qquad c_3(r^2,1)=\frac{4}{5}\,, \tag{11.1}$$

where again $r=(\sqrt{5}-1)/2$. This leads to the following table of possible 1-particle rational theories in $3D$ which is again very likely to be exhaustive, but we cannot prove it.

The interacting bosonic theory with $c_{\rm qs}=7/8$ was suggested to be the 3D Ising model in section 8. The $c_{\rm qs}=4/5$ theory appears to be related to the $O(N)$ sigma model, which at large N has $c_{\rm qs}=\frac{4}{5}N$ [7]. The method for computing $c_{\rm qs}$ in the latter work is just the effective action at large N, and is thus different than our methods.

statistics	$c_{\rm qs}$	g	h
fermion	$3/4$	0	0
boson	1	0	0
boson	$7/8$	$\log 2$	-1
boson	$4/5$	$-\log r^2$	-2

Table 2: Rational 1-particle theories in 3D.

11.2 Multi-particle RSPE's in $3D$

For the remaining RSPE's presented in this paper we only reconstruct the particle spectrum, gap equation, and interaction parameters h_{ab}, since as previous stated we do not have the tools necessary to further reconstruct the theory. There are a number of generic features of these multi-particle RSPE's. They are generally of mixed statistics, i.e. involve both bosons and fermions. One can also check that in these examples, the interactions always reduce $c_{\rm qs}$ from it's free field value, i.e. $c_{\rm qs} < N_b + (1 - 1/2^d)N_f$, where $N_{b,f}$ are the numbers of bosons and fermions. Below, we present this feature with the parameter $r = \frac{c_{\rm qs}}{c_{\rm free}}$. However this is not generally the case, as interactions in fermionic theories can increase $c_{\rm qs}$, the extreme-fermion being the limiting case. (See eq. (13.1).) RSPE's appear to be more and more scarce as one moves up in dimension. This is not surprising since conformal field theories are also expected to be more scarce since there is no known underlying Virasoro algebra with a rich representation theory.

Let us reconstruct the following $c_{\rm qs} = \frac{13}{6}$ RSPE:

$$2c_3\left(\frac{1}{3}, 1\right) + c_3\left(\frac{1}{3}, -1\right) = \frac{13}{6}\,. \tag{11.2}$$

It contains 2 bosonic particles with $z_1 = 1/3$ and one fermionic particle with $z_2 = 1/3$. Here $r = 26/33$. The gap equations then read

$$z_1 = (1 - z_1)^{-2h_{11}}(1 + z_1)^{h_{12}}\,, \qquad z_2 = (1 + z_2)^{h_{22}}(1 - z_1)^{-2h_{21}}\,. \tag{11.3}$$

The solution is $h_{ab} = -1$, $\forall\, a, b$.

Consider next the $c_{\rm qs} = \frac{11}{4}$ RSPE with 3 bosons and 2 fermions with $r = 11/18$.

$$3c_3(x, 1) + 2c_3(x, -1) = \frac{11}{4}\,, \qquad x \equiv 3 - 2\sqrt{2}\,. \tag{11.4}$$

Here the gap equation reads:

$$z_1 = (1 - z_1)^{-3h_{11}}(1 + z_2)^{2h_{12}}\,, \qquad z_2 = (1 + z_2)^{2h_{22}}(1 - z_1)^{-3h_{21}}\,. \tag{11.5}$$

The novel feature here is that there is a one-parameter family of solutions to h:

$$h_{21} = h_{11}\,, \qquad h_{12} = h_{22} = \frac{3h_{11}\log(1 - x) + \log x}{2\log(1 + x)}\,. \tag{11.6}$$

This suggests some exactly marginal directions since the h's are required to be RG invariants.

11.3 Multi-particle RSPE's in $4D$

In 4D there are no known RSPE's beyond the free fermion, free boson and extreme-fermion. We present only two examples of multiparticle RSPE's.

Here is an RSPE with $c_{\rm qs} = 179/2$ and $r = 716/721$:

$$42\, c_4 \left(\frac{1}{2}, 1\right) + c_4 \left(\frac{1}{8}, -1\right) + 54\, c_4(2, -1) = \frac{179}{2}\,. \tag{11.7}$$

It consists of 42 bosons and 55 fermions. Here there is a 3 parameter family of interaction parameters that give the gap parameters z:

$$\begin{aligned} h_{12} &= 14 h_{11}\,, & h_{21} &= \frac{1}{14}(h_{22} - 1)\,, & h_{13} &= -\frac{14}{27} h_{11} \\ h_{31} &= \frac{1}{42}(1 - 81 h_{33})\,, & h_{23} &= -\frac{1}{27} h_{22}\,, & h_{32} &= -27 h_{33}\,. \end{aligned} \tag{11.8}$$

We leave the following $c_{\rm qs} = 4393/8$, $r = 4393/4513$ example as an exercise:

$$264\, c_4(\chi, 1) + 15\, c_4(\chi^2, -1) + 4\, c_4(\chi^3, -1) + 324\, c_4 \left(\frac{1}{\chi}, -1\right) = \frac{4393}{8}\,, \tag{11.9}$$

where $\chi \equiv 2 - \sqrt{3}$.

12. Fractional statistics and imaginary chemical potential

12.1 General arguments

In the formalism we have developed so far, the bosonic or fermionic statistics ($s = \pm 1$) has played a crucial role. It is well-known that in 2D and 3D, fractional statistics particles, i.e. anyons, are also possible [53]. In this section we argue that this possibility can be incorporated with an imaginary chemical potential. The connection between anyons and an imaginary chemical potential has previously been suggested in the context of the 3D Gross-Neveu model [52].

Let us model our discussion based on the 3D case, though we will also present results in 2D. The standard way to obtain anyons is to add to the action a topological, or Hopf, term:

$$S \to S + i\vartheta\, S_{\rm top}\,. \tag{12.1}$$

The simplest case, which we will refer to as the free anyon, corresponds to S being free charged bosons or fermions with U(1) current J^μ and charges $Q = \pm 1$. Here $S_{\rm top}$ is the Chern-Simons term where the U(1) gauge field A_μ is manufactured from the U(1) matter current $J^\mu = \epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha$. The statistical parameter ϑ is normalized so that the spin of the particle is $\vartheta/2\pi$.

At finite temperature, space-time has the topology of $R^d \otimes S^1$, where S^1 is the one dimensional circle. Let $\mathcal{C}$ denote the configuration space of the field theory; for example for the $O(3)$ sigma model, $\mathcal{C} = S^2$. At finite temperature the relevant homotopy group is the one classifying maps from $R^d \otimes S^1 \to \mathcal{C}$ which we will denote as π_T.

Let us assume $\pi_T = \mathbb{Z}$ and the Hopf term $S_{\rm top}$ is ± 1 for the particles of U(1) charge $Q = \pm 1$. This implies that the particles contribute $\pm i\vartheta$ to $\log Z$. Comparing with eq. (5.2), this corresponds to a chemical potential $\mu = \pm i\vartheta T$ where T is the temperature. More generally, let μ_a denote the imaginary chemical potential which modifies the statistics of the a-th particle with statistical parameter θ_a:

$$\mu_a = i\vartheta_a T\,. \tag{12.2}$$

The important feature of this *statistical chemical potential* is that μ_a/T remains finite at any temperature, which is a consequence of the fact that $S_{\rm top}$ is finite. This leads to a finite contribution to the thermal gaps $g_a = -i\vartheta_a$. In terms of the gap variables z_a the gap equation reads:

$$z_a = e^{i\vartheta_a} \prod_b (1 - s_b z_b)^{-h_{ab} s_b}\,, \tag{12.3}$$

where as before $s_a = \pm 1$ corresponds to whether the original undressed particles were fermions or bosons.

If the particles are free, $h_{ab} = 0$ and $z_a = e^{i\vartheta_a}$, i.e. the thermal gaps are purely statistical. Let us denote by $\widetilde{c}_D(\vartheta, s)$ the value of $c_{\rm qs}$ in D-dimensions where $s = 1$ ($s = -1$) corresponds to ϑ-statistically dressed charged free bosons (fermions) with $\vartheta_\pm = \pm\vartheta$. From the results of section 7, we propose the following formula for these "free anyons":

$$\widetilde{c}_D(\vartheta, s) = \frac{s}{\zeta(D)} \left({\rm Lr}_D(s e^{i\vartheta}) + {\rm Lr}_D(s e^{-i\vartheta})\,.\right) \tag{12.4}$$

(For z on the unit circle, $|z| = 1$ in the formula (7.17) so that ${\rm Lr}(z) = {\rm Li}(z)$.)

12.2 Free anyons in 2D

A non-trivial check of eq. (12.4) is available in 2D due to the remarkable dilogarithm identities for z on the unit circle:

$$\widetilde{c}_2(\vartheta, 1) = 2\left(1 - \frac{3\vartheta}{\pi}\left(1 - \frac{\vartheta}{2\pi}\right)\right), \qquad 0 < \vartheta < 2\pi \tag{12.5}$$

$$\widetilde{c}_2(\vartheta, -1) = 1 - 12\left(\frac{\vartheta}{2\pi}\right)^2 \qquad -\pi < \vartheta < \pi\,. \tag{12.6}$$

The arguments leading to eq. (12.4) assumed the underlying field theory was a free theory of charged fractional statistics particles. In 2D the spin σ of a field can be defined by monodromy property:

$$\psi(e^{2\pi i} z, e^{-2\pi i}\overline{z}) = e^{2\pi i\sigma}\psi(z, \overline{z})\,, \tag{12.7}$$

where $z, \overline{z}$ are the euclidean light-cone space-time variables, $z = x + iy, \overline{z} = x - iy$. The simplest free theories of anyons in 2D are the so-called $\beta - \gamma$ systems [54] which were first studied in connection with the world-sheet ghosts in string theory. The theories are defined by the action

$$S = \frac{1}{2\pi} \int d^2x \;\left(\beta\partial_{\overline{z}}\gamma + \overline{\beta}\partial_z\overline{\gamma}\right), \tag{12.8}$$

where the spin of (β,γ) is $(\sigma, 1-\sigma)$ and the opposite sign for $(\overline{\beta},\overline{\gamma})$. The fields can be quantized as bosons or fermions, the latter usually being referred to as a $b-c$ system.

Consider first the bosonic $\beta-\gamma$ system. The central charge is known to be [54]:

$$c_{\beta-\gamma} = 2(6\sigma^2 - 6\sigma + 1)\,. \tag{12.9}$$

Due to the identity eq. (12.5), this corresponds precisely to the statistically dressed boson, i.e. $c_{\beta-\gamma} = \widetilde{c}_2(\vartheta, 1)$ with $\vartheta/2\pi = \sigma$.

Consider next the case where $\beta-\gamma$ are quantized as fermions, usually denoted $b-c$. The central charge is known to be:

$$c_{b-c} = -2(6\sigma^2 - 6\sigma + 1)\,. \tag{12.10}$$

Since the undressed particles are already fermions, by the same reasoning as before, this should correspond to a statistically dressed fermion with $\vartheta/2\pi = \sigma - 1/2$; it is easily confirmed that $\widetilde{c}_2(\vartheta, -1) = c_{b-c}$ using eq. (12.6).

It is somewhat remarkable that we have been able to obtain the above central charges simply from a statistical thermal gap. We point out that equality of $c_{\beta-\gamma}, c_{b-c}$ with $\widetilde{c}_2(\vartheta, s)$ is only valid for the range of ϑ indicated in eqs. (12.5), (12.6) since $\widetilde{c}_2(\vartheta, s)$ is a periodic function of ϑ whereas $c_{\beta-\gamma}, c_{b-c}$ are quadratic.

12.3 Rational free anyons in 3D

As the equations (12.5), (12.6) show, when $\vartheta/2\pi$ is rational then $\widetilde{c}_2(\vartheta, s)$ is rational. In 3D, there is no analog of these equations and $\widetilde{c}_3(\vartheta, s)$ is generally irrational. However for some special values of ϑ, $\widetilde{c}_3$ turns out to be rational.

Anyons of spin 1/3 or 1/6 are examples of rational theories:

$$\widetilde{c}_3\left(\frac{\pi}{3}, 1\right) = -\widetilde{c}_3\left(\frac{2\pi}{3}, -1\right) = \frac{2}{3}\,, \qquad \widetilde{c}_3\left(\frac{\pi}{3}, -1\right) = -\widetilde{c}_3\left(\frac{2\pi}{3}, 1\right) = \frac{8}{9}\,. \tag{12.11}$$

An interesting feature of this case is that these gap parameters can be obtained from an interacting theory with no statistical chemical potential. Consider for example the bosonic theory with $\widetilde{c}_3 = 2/3$. There are two particles with gap parameters $z_1 = e^{i\pi/3}, z_2 = e^{-i\pi/3}$. These are the solution to the bosonic gap equation

$$z_1 = (1-z_1)^{-h_{11}}(1-z_2)^{-h_{12}}\,, \qquad z_2 = (1-z_1)^{-h_{21}}(1-z_2)^{-h_{22}} \tag{12.12}$$

with real interaction parameters $h_{11} = h_{22} = 1/2$, $h_{12} = h_{21} = -1/2$. This observation suggests that anyons can arise in an ordinary interacting theory without inclusion of a Hopf term.

Another example we have found is for particles of spin 1/4:

$$\widetilde{c}_3\left(\frac{\pi}{2}, -1\right) = -\widetilde{c}_3\left(\frac{\pi}{2}, 1\right) = \frac{3}{16}\,. \tag{12.13}$$

It remains unknown to us whether there are other values of ϑ that correspond to rational theories.

13. Phase structure

In this section we reintroduce the chemical potential and address a basic issue: for given interaction parameter h and chemical potential μ, is there a fixed point with a well-defined $c_{\rm qs}$? Generally speaking, μ is related to the density of particles, where $\mu > 0$ ($\mu < 0$) corresponds to increased (decreased) density. This issue has largely been ignored even in the 2D integrable TBA approach.

Given the interaction parameters h and the solutions to the gap equation z, $c_{\rm qs}$ as given in eq. (7.19) is not necessarily real, or even well-defined, and this could signify something like a phase transition. In this section we explore this issue for 1-particle theories, where a real $c_{\rm qs}$ requires a real z. Of course as the last section shows, real $c_{\rm qs}$ can arise for complex z in multi-particle theories.

The functions ${\rm Lr}_{d+1}(z)$ are real for $-\infty < z < 1$. This implies the following physical ranges of z:

$$\begin{aligned} &\textbf{bosons}: \qquad -\infty < z < 1, \qquad -2\left(1-\frac{1}{2^d}\right) < c_{\rm qs}(z) < 1 \\ &\textbf{fermions}: \qquad -1 < z < \infty, \qquad -1 < c_{\rm qs}(z) < 2\left(1-\frac{1}{2^d}\right). \end{aligned} \tag{13.1}$$

In both cases $c_{\rm qs} < 0$ corresponds to $z < 0$.

Let us write the gap equation as

$$f^{(h)}_{\mp}(z) \equiv \frac{z}{(1\mp z)^{\mp h}} = e^{\mu/T} \tag{13.2}$$

where μ is the chemical potential. We are only considering fixed points here, so μ must be proportional to T for it to have any affect on $c_{\rm qs}$. Given h, μ, one can graphically obtain the gap z by first plotting the function $f^{(h)}_{\mp}(z)$ as a function of z for the ranges in eq. (13.1). A given μ, by eq. (13.2), is a point on the y-axis. Thus, the intersection of a horizontal line crossing the y-axis at $e^{\mu/T}$ with the $f^{(h)}$ curves determines the gap z. There are several cases depending on h and whether the particles are bosons or fermions.

Bosons with $h < 0$. This case is shown in figure 2. One sees that for any real μ there is a single solution with $0 < z < 1$. Solutions with a negative z, on the negative y-axis, require the chemical potential to have an imaginary part $Im(\mu) = \pi T$. As explained in the last section, this can correspond to a statistical chemical potential with $\vartheta = \pi$. In this regime there are two solutions to z if $h < -1$, one solution if $-1 < h < 0$, and no solution if the real part of μ is too large. Since in all cases a negative z requires an imaginary part to the chemical potential, throughout this section by "complex chemical potential" we mean one where specifically $Im(\mu) = \pi T$.

Bosons with $h > 0$. This case is shown in figure 3. Here, for real chemical potential, there is no solution unless μ is negative and

$$\frac{\mu}{T} < -\left((h+1)\log(h+1) - h\log h\right) \tag{13.3}$$

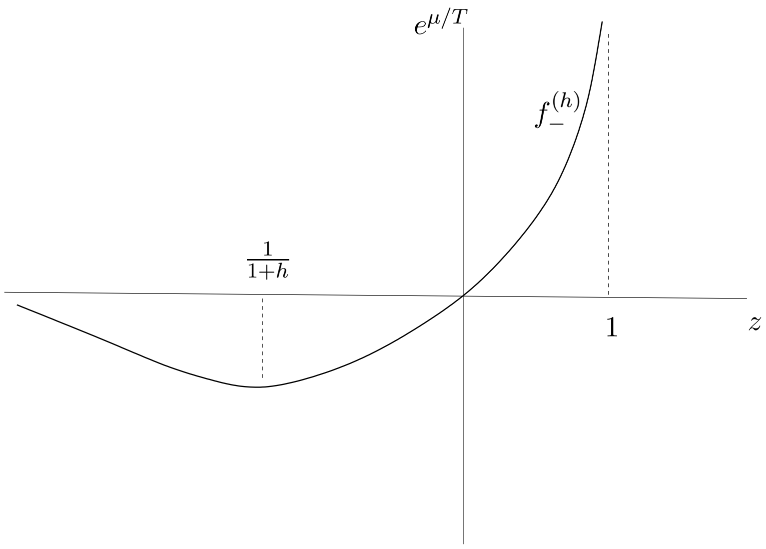

Figure 2: Bosons with $h < 0$.

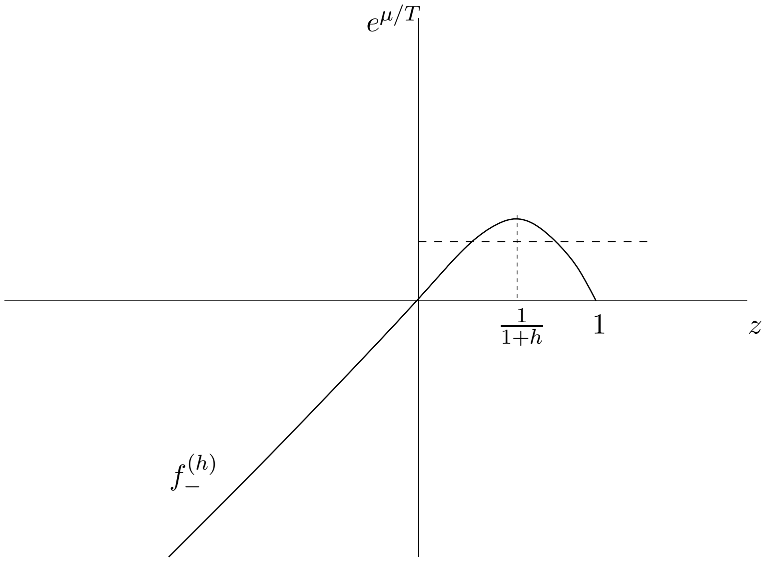

Figure 3: Bosons with $h > 0$.

in which case there are two solutions with $0 < z < 1$. Thus there is no solution for zero chemical potential. The horizontal line displaying the two solutions is rendered in the figure. This implies that if the gas is too dense, there is no fixed point. The latter means for instance that the black body formula breaks down. This could signify a kind of Bose-Einstein condensation. This explains for example why one cannot obtain a well-defined UV limit for the bosonic version of the sinh-Gordon model, as explored in [47], unless one lowers the density. As the density is decreased, there are two possible $c_{\rm qs}$. It is not entirely clear what the significance is of these two solutions. If they are both possible in the same model, then this suggests there are two phases and which one is reached depends on the details of the RG flow. On the other hand there is always a solution with $z < 0$ with a complex chemical potential.

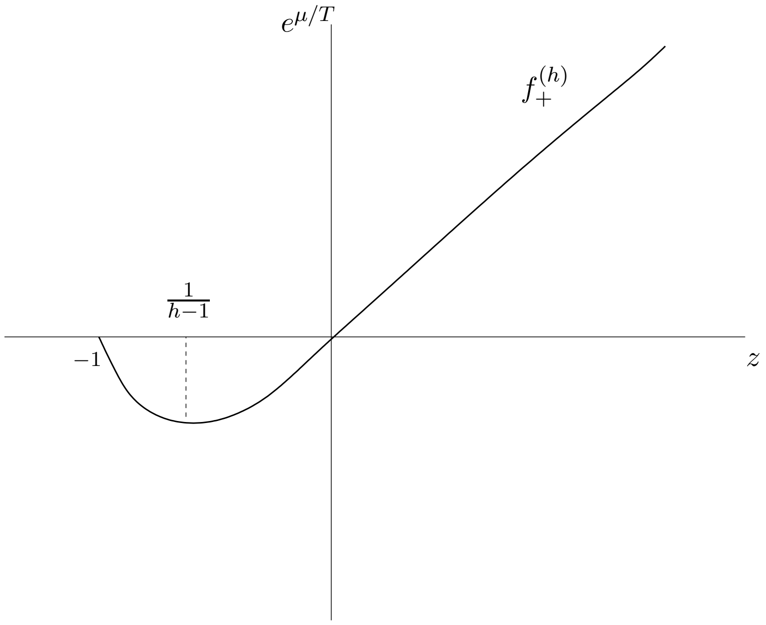

Figure 4: Fermions with $h < 0$.

Fermions with $h < 0$. See figure 4. For any real μ there is a single solution with $0 < z < \infty$. With complex chemical potential there are two solutions if $\mathrm{Re}(\mu)$ is not too large.

Fermions with $h > 0$. See figure 5. For $0 < h < 1$, and real negative chemical potential there is one solution with $0 < z < \infty$. The sinh-Gordon model is the special case $h = 1$ where there is one solution with $z = \infty$. For $h > 1$ there is no solution unless

$$\frac{\mu}{T} < -\left(h \log h - (h-1)\log(h-1)\right) \tag{13.4}$$

in which case there are two solutions. Finally with complex chemical potential there is always a unique solution with $-1 < z < 0$.

14. Closing remarks

The quasi-particle form of the finite temperature partition function and the integral gap equation is a new formulation of finite temperature field theory that appears to be well suited for capturing certain phenomenon in a clear fashion. Though the method is an approximation in general, and the nature of this approximation needs further clarification, we believe that it captures the essential mechanism that leads to non-trivial fixed points: interactions lead to thermal gaps that are proportional to the temperature, $\Delta = gT$, and c_{qs} is given in terms of some very special combinations of polylogarithmic functions eq. (7.17) of the complex variable $z = e^{-g}$. The spectacular properties of these functions $\mathrm{Lr}_n(z)$ lead us to believe that this is a general property, i.e. that any non-trivial fixed point has c_{qs} given by the expression (7.23). The classification of rational conformal theories in any dimension

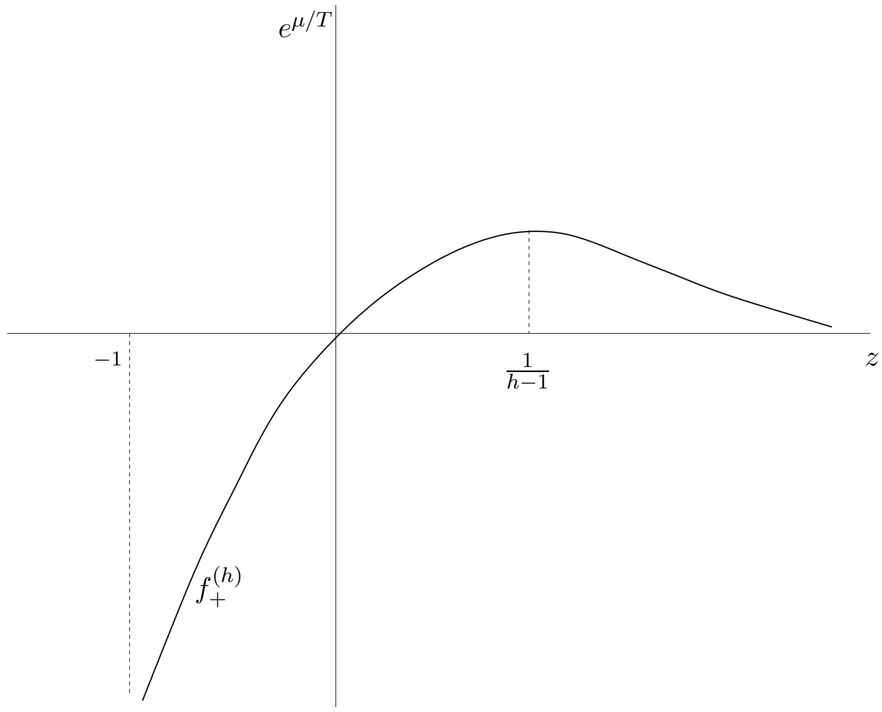

Figure 5: Fermions with $h > 0$.

is then essentially mapped onto the problem of the construction of polylogarithmic ladders in mathematics. It would be very interesting to use this idea to explain some of the rational values of $c_{\rm qs}$ obtained for superconformal gauge theories using string techniques [55].

The thermal gaps are related to the fundamental interaction parameters of the theory by the algebraic gap equation (7.22). There may be other methods that lead to the algebraic gap equation besides the integral gap equation proposed in section 5, such as the finite size effective potential [7], and this is important to explore.

One of the main questions left unanswered in this paper is how the structure of the stress-tensor form factors can lead to finite interaction parameters h_{ab} that are RG invariants. In 2D this is well-understood using integrability and the relation of the form factors to the exact S-matrix. Understanding this is clearly necessary for further progress on specific models.

We have touched only briefly on the possible applications. Further work is clearly needed on some of the suggestions made in this paper. For example, the proposals for the 3D Ising model and the free energy of anyons in 3D should be compared with numerical simulations of the free energy.

To phrase properties of the equation of state of matter in cosmology in terms of the c-function $C_{\rm qs}(T)$ appears to be fruitful. As we argued in section 4, this leads to the bounds $-1 < w < -1/3$ for dark energy, but perhaps more importantly we have related the temperature dependence of w to the RG flow of $C_{\rm qs}$, and thus "c-theorems" become relevant. For instance, from this perspective the fate of the accelerated expansion depends on whether the dark energy has an infra-red fixed point.

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